Interactions between quantum probability and operator space theory

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0 Introduction

It is well-known that probabilistic methods are important methods in Banach space theory. As operator spaces are quantized Banach spaces, one would naturally expect that quantum probability should play a non negligible role in the young operator space theory. This is indeed the case. In fact, quantum probability and operator space theory are intimately related and there exist many interactions between them. For instance, the recent remarkable development of noncommutative martingale inequalities is directly influenced and motivated by operator space theory. In particular, the establishment of the noncommutative Doob maximal inequality by Junge [J1] is inspired by Pisier's theory of vector-valued noncommutative L_p-spaces. On the other hand, the noncommutative Burkholder/Rosenthal inequalities can be used to determine the linear structure of symmetric subspaces of noncommutative L_p-spaces (see [JX]). The recent works of Pisier/Shlyakhtenko [PS] on the operator space Grothendieck inequality and of Junge [J2] on the complete embedding of OH into a noncommutative L_1 are two more beautiful examples of illustration of these interactions. Certain Khintchine type inequalities are key ingredients in both works.

It is clear today that quantum probability is of increasing importance in operator space theory. We will try to convince the reader of this in this course by presenting a very brief aspect of interactions between the two theories. Our presentation is around Khintchine type inequalities. In consequence, noncommutative L_p-spaces are at the heart of these lectures.

This course can be divided into three parts. The first one gives a brief introduction to operator space theory. We start with a short discussion on completely positive maps between C*-algebras in order to prepare the introduction to operator spaces and completely bounded maps. Our introduction to operator spaces begins with concrete operator spaces, i.e., those which are closed subspaces of $B(H)$ and the Haagerup-Paulsen-Wittstock factorization theorem for completely bounded maps, which should be understood together with its predecessor, Stinespring's factorization theorem for completely positive maps. We then pass to Ruan's fundamental characterization of abstract operator spaces. It is Ruan's theorem that allows us to do basic operations on operator spaces such that duality, quotient and interpolation. Meanwhile, some important examples of operator spaces are given, including column space C, row space R and Pisier's operator Hilbert space OH. This first part ends with an outline of Pisier's vector-valued noncommutative L_p-spaces. We concentrate here only on Schatten classes.

The second part is of quantum probabilistic character. The main object of this part is various noncommutative Khintchine inequalities. It becomes clear nowadays that Khintchine type inequalities are of paramount importance in operator space theory and more generally in noncommutative analysis. The inequalities we present are those for Rademacher variables, free group generators,

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Voiculescu's semicircular systems and Shlyakhtenko's generalized circular systems. The proofs of these inequalities in noncommutative L_p-spaces are often quite technical and tricky. But what will be needed in the third part concerns only the case $p = 1$, which is often easier. We should point out that the von Neumann algebra generated by a generalized circular system is of type III, which turns our presentation of noncommutative L_p-spaces somehow more complicated. This is, unfortunately, unavoidable in view of the complete embedding of OH into a noncommutative L_1 presented later.

The short third part is devoted to Junge's complete embedding of OH into a noncommutative L_1. We first present OH as a quotient of a subspace of $C \oplus R$ (a theorem of Pisier), which is given by the graph of a closed densely defined operator on ℓ_2. We then prove that any such graph embeds completely isomorphically into a noncommutative L_1-space.

1 Completely positive maps

This section gives a brief discussion on completely positive maps between C*-algebras. Our references for operator algebras are [KR] and [T]. Let us fix some notations used throughout this course:

- $B(H)$ denotes the space of all bounded linear operators on a complex Hilbert space H.
- A often denotes a C*-algebra; so A can be regarded as a C*-subalgebra of $B(H)$ for some H. A_+ denotes the positive cone of A.
- M denotes a von Neumann algebra and M_* the predual of M.
- $\mathbb{M}_n$ denotes the algebra of complex $n \times n$ matrices; so $\mathbb{M}_n$ can be identified with $B(\ell_2^n)$.
- $\mathbb{M}_n(A)$ denotes the algebra of $n \times n$ matrices with entries in A. This is again a C*-algebra. If $A \subset B(H)$, then $\mathbb{M}_n(A)$ is a C*-subalgebra of $\mathbb{M}_n(B(H)) \simeq B(\ell_2^n(H))$.
- ℓ_p denotes the ℓ_p-space of complex sequences (x_k) such that $(\sum_k |x_k|^p)^{1/p} < \infty$, $1 \le p \le \infty$ (with the usual modification for $p = \infty$). The n-dimensional version of ℓ_p is denoted by ℓ_p^n.

Definition 1.1 Let $u : A \to B$ be a linear map between two C*-algebras.

i) u is called *positive* if $u(A_+) \subset B_+$.

ii) u is called *completely positive* (*c.p.* for short) if u_n is positive for every $n \ge 1$, where $u_n = \mathrm{id}_{\mathbb{M}_n} \otimes u : \mathbb{M}_n(A) \to \mathbb{M}_n(B)$ is defined by $u_n((x_{ij})) = (u(x_{ij}))$.

Remark 1.2 It is easy to check that a positive map $u : A \to B$ is automatically continuous. If in addition A is unital, then $\|u\| = \|u(1)\|$ (see [Pa]).

Examples:

1) **Homomorphisms.** Every homomorphism $\pi : A \to B$ is c.p.. By *homomorphism* we mean a linear map satisfying: $\pi(xy) = \pi(x)\pi(y)$ and $\pi(x^*) = \pi(x)^*$ for all $x, y \in A$. Recall that a homomorphism π is necessarily contractive, i.e., $\|u\| \le 1$. If in addition π is injective, then π is isometric.

2) **Multiplications.** Let $a \in A$ and define $C_a : A \to A$ by $C_a(x) = a^*xa$. Then C_a is c.p. and $\|C_a\| = \|a^*a\| = \|a\|^2$. More generally, if H and K are two Hilbert spaces and $a : K \to H$ a bounded operator, then $C_a : B(H) \to B(K)$ defined by $C_a(x) = a^*xa$, is c.p..

The classical theorem of Stinespring states that any c.p. map is the composition of two maps of the previous types.

Theorem 1.3 (**Stinespring's factorization**)
Let A be a C-algebra and B a C*-subalgebra of $B(K)$. Let $u : A \to B$ be c.p.. Then there are a Hilbert space H, a representation $\pi : A \to B(H)$ (i.e., a homomorphism) and a bounded operator $a : H \to K$ such that*

$$u(x) = a^*\pi(x)a, \quad \forall\, x \in A.$$

Namely, $u = C_a \circ \pi$.

We refer to [T] for the proof of this theorem. We deduce immediately the following

Corollary 1.4 *If $u : A \to B$ is c.p., then*

$$\|u\|_{cb} \overset{\text{def}}{=} \sup_n \|u_n : \mathbb{M}_n(A) \to \mathbb{M}_n(B)\| < \infty.$$

Namely, u is completely bounded. Moreover, $\|u\|_{cb} = \|u\|$.

Exercices:

1) Prove that a positive map is automatically continuous.

2) Let A be a C*-algebra. Prove that $x = (x_{ij})_{i,j} \in \mathbb{M}_n(A)$ is positive iff x is a sum of matrices of the form $(a_i^* a_j)_{i,j}$ with $a_1, ..., a_n \in A$.

3) Let A and B be C*-algebras with B commutative. Prove that every positive map $\varphi : A \to B$ is automatically c.p..

2 Concrete operator spaces and completely bounded maps

We consider in this section closed subspaces of $B(H)$ and completely bounded maps between them. The analogue for the present setting of the Stinespring factorization given in the previous section is the Haagerup-Paulsen-Wittstock factorization for completely bounded maps, which is fundamental in the theory. The references for this and next sections are [ER], [Pa] and [P3].

Definition 2.1 A (*concrete*) *operator* space is a closed subspace E of $B(H)$ for some Hilbert space H.

Let E be a Banach space, and let B_{E^*} denote the unit ball of the dual E^* of E. B_{E^*} becomes a compact topological space when equipped with the w*-topology. Then $E \subset C(B_{E^*})$ isometrically. More precisely, given $x \in E$ define $\widehat{x} : B_{E^*} \to \mathbb{C}$ by $\widehat{x}(\xi) = \xi(x)$. Then $x \mapsto \widehat{x}$ establishes an isometry from E into $C(B_{E^*})$. But now $C(B_{E^*})$ is a commutative C*-algebra, so $C(B_{E^*}) \subset B(H)$ for some H. In this way, any Banach space is an operator space. However, according to the preceding definition, an operator space E is given together with an embedding of E into a $B(H)$. More precisely, an operator space is a pair of a Banach space E and an embedding of E into some $B(H)$. On the other hand, $B(H)$ admits a natural matricial structure: $\mathbb{M}_n(B(H)) = B(\ell_2^n(H))$ for every $n \in \mathbb{N}$. This should be reflected in E too. In particularly, the "admissible" morphisms in the category of operator spaces should respect this matricial structure.

Matricial structure. Let $E \subset B(H)$ be an operator space. Then E inherits the matricial structure of $B(H)$ by virtue of the embedding $\mathbb{M}_n(E) \subset \mathbb{M}_n(B(H))$. More precisely, let $\mathbb{M}_n(E)$ be the space of $n \times n$ matrices with entries in E. Then $\mathbb{M}_n(E)$ is equipped with the norm induced by that of $B(\ell_2^n(H))$.

Definition 2.2 Let $E \subset B(H)$ and $F \subset B(K)$ be two operator spaces. Let $u : E \to F$ be a linear map.

i) u is called *completely bounded* (*c.b.* for short) if

$$\|u\|_{cb} = \sup_{n\geq 1} \|u_n\| < \infty,$$

where $u_n = \mathrm{id}_{\mathbb{M}_n} \otimes u : \mathbb{M}_n(E) \to \mathbb{M}_n(F)$ is defined by $u_n((x_{ij})) = (u(x_{ij}))$. $CB(E, F)$ denotes the space of all c.b. maps from E to F.

ii) u is called a *complete isomorphism* if u is a c.b. bijection and u^{-1} is also c.b..

iii) u is called *completely isometric* if u_n is isometric for every n. If u is a bijection and both u and u^{-1} are completely isometric, u is called a *complete isometry.*

Examples:

1) **C*-algebras.** Let A be a C*-algebra. Then A is a C*-subalgebra of some $B(H)$; so A is an operator space. The resulting matricial structure on A is called the *natural operator structure* of A. In particular, any von Neumann algebra has its natural operator structure. Note that this natural operator space structure of A is independent, up to complete isometry, of particular representation of A as C*-subalgebra of $B(H)$ for a faithful representation is complete isometric.

2) **Minimal structure.** Let E be a Banach space. Then we have the isometric embedding $E \subset C(B_{E^*})$, which turns E an operator space, denoted by $\min(E)$. This operator space structure is called the *minimal structure* on E. The adjective "minimal" means that this structure induces the least matricial norms on $\mathbb{M}_n(E)$. Indeed, assume that E itself is an operator space and consider also the associated minimal operator space $\min(E)$. Then by Proposition 2.3 below, the identity map on E induces a complete contraction from E to $\min(E)$. Namely, for any n and any $x \in \mathbb{M}_n(E)$

$$\|x\|_{\mathbb{M}_n(\min(E))} \leq \|x\|_{\mathbb{M}_n(E)}\,.$$

3) **Maximal structure.** Similarly, we can introduce a maximal operator structure on a Banach space E. Let Φ be the family of all pairs $\varphi = (u_\varphi, H_\varphi)$ with H_φ a Hilbert space and $u_\varphi : E \to B(H_\varphi)$ a contraction. Let $\mathcal{H}$ be the Hilbert space direct sum of the H_φ:

$$\mathcal{H} = \bigoplus_{\varphi\in\Phi} H_\varphi\,.$$

Then $B(\mathcal{H})$ is the C*-algebra direct sum of the $B(H_\varphi)$:

$$B(\mathcal{H}) = \bigoplus_{\varphi\in\Phi} B(H_\varphi)\,.$$

Define $J : E \to B(\mathcal{H})$ by $J(x) = \big(u_\varphi(x)\big)_{\varphi\in\Phi}$. It is clear that J is isometric. This yields an isometric embedding of E into $B(\mathcal{H})$. Identifying E and $J(E) \subset B(\mathcal{H})$, we turn E an operator space. This operator space structure is called the *maximal structure* of E and denoted by $\max(E)$. By definition, one sees that if E itself is an operator space, then the identity map $\mathrm{id}_E : \max(E) \to E$ is completely contractive.

4) **Column and row spaces.** Let (e_{ij}) denote the standard matrix units of $B(\ell_2)$. Define

$$C = \overline{\mathrm{span}}(e_{i1},\ i \geq 1) \quad \text{and} \quad R = \overline{\mathrm{span}}(e_{1j},\ j \geq 1).$$

Thus we get two operator spaces $C \subset B(\ell_2)$ and $R \subset B(\ell_2)$. Note that C (resp. R) is identified as the first column (resp. row) subspace of $B(\ell_2)$. The operator space structures of C and R are easily determined. Let $x \in \mathbb{M}_n(C), y \in \mathbb{M}_n(R)$ and write

$$x = \sum_i x_i \otimes e_{i1}, \quad y = \sum_i y_i \otimes e_{1i}, \quad x_i,\ y_i \in \mathbb{M}_n.$$

Then

$$\|x\|_{\mathbb{M}_n(C)} = \Big\| \sum_i x_i^* x_i \Big\|^{1/2}, \quad \|y\|_{\mathbb{M}_n(R)} = \Big\| \sum_i y_i y_i^* \Big\|^{1/2}.$$

The n-dimensional versions of C and R are denoted by C^n and R^n, respectively. As Banach spaces both C and R are isometric to ℓ_2 via the identification $e_{i1} \sim e_{1i} \sim e_i$, where (e_i) denotes the canonical basis of ℓ_2. We will see at the end of this section that they are not completely isomorphic.

Proposition 2.3 *Let $E \subset B(H)$ be an operator space and A a commutative C*-algebra. Then any bounded map $u : E \to A$ is automatically c.b. and $\|u\|_{cb} = \|u\|$.*

Proof. Let us first consider the case where $A = \mathbb{C}$. Let $u : E \to \mathbb{C}$ be a continuous linear functional with $\|u\| \leq 1$. For any $n \in \mathbb{N}$ we must show that $u_n : \mathbb{M}_n(E) \to \mathbb{M}_n$ is a contraction. To this end let $x \in \mathbb{M}_n(E)$, and let $\alpha, \beta \in \ell_2^n$ (α and β being viewed as column matrices). Then

$$\begin{aligned} \big|\langle u_n(x)\alpha,\ \beta\rangle\big| &= \big| \sum_{i,j=1}^n u(x_{ij})\alpha_j \bar{\beta}_i \big| = \big| u\Big(\sum_{i,j=1}^n x_{ij}\alpha_j \bar{\beta}_i \Big) \big| \\ &\leq \big\| \sum_{i,j=1}^n x_{ij}\alpha_j \bar{\beta}_i \big\| = \big\| \beta^* x \alpha \big\| \leq \|\beta\|\,\|x\|\,\|\alpha\|. \end{aligned}$$

Taking the supremum over all x, α, β in the respective unit balls, we deduce that $\|u_n\| \leq 1$. Thus u is a complete contraction.

The general case can be easily reduced to the previous one. Indeed, since A is commutative, we can assume $A = C_0(\Omega)$ for some locally compact topological space Ω, where $C_0(\Omega)$ denotes the C*-algebra of all continuous functions on Ω which tend to zero at infinity. Then $\mathbb{M}_n(A) = C_0(\Omega; \mathbb{M}_n)$, the C*-algebra of continuous functions from Ω to $\mathbb{M}_n$ which vanish at infinity. The norm of an element $y = (y_{ij}) \in C_0(\Omega; \mathbb{M}_n)$ is given by

$$\|y\| = \sup_{\omega \in \Omega} \big\| \big(y_{ij}(\omega)\big) \big\|_{\mathbb{M}_n}.$$

Now let $u : E \to A$ be bounded. Then

$$\|u_n\| = \sup \big\{ \big\| \big(u(x_{ij})(\omega)\big) \big\|_{\mathbb{M}_n} \ :\ \omega \in \Omega,\ x \in \mathbb{M}_n(E),\ \|x\| \leq 1 \big\}.$$

It follows that

$$\|u\|_{cb} = \sup_{\omega \in \Omega} \|\delta_\omega \circ u\|_{cb},$$

where $\delta_\omega : C_0(\Omega) \to \mathbb{C}$ is the evaluation at ω: $\delta_\omega(f) = f(\omega)$. We are thus reduced to the one dimensional case. □

Prototypical examples of c.b. maps:

1) **Homomorphisms between C*-algebras**. These maps are c.b. for they are c.p.. Moreover, they are completely contractive. Note also that an injective homomorphism is completely isometric.

2) **Multiplications by bounded operators**. Given $a \in B(H)$ define

$$L_a : B(H) \to B(H),\ x \mapsto ax \quad \text{and} \quad R_a : B(H) \to B(H),\ x \mapsto xa.$$

Then L_a and R_a are c.b. and

$$\|L_a\|_{cb} = \|R_a\|_{cb} = \|a\|.$$

Thus if $E \subset B(H)$, then $L_a\big|_E$ and $R_a\big|_E$ are also c.b..

The following theorem asserts that any c.b. map is the composition of a homomorphism, a left multiplication and a right multiplication. This is the c.b. analogue of Stinespring's factorization. We refer to [ER] and [P3] for the proof.

Theorem 2.4 (**Haagerup-Paulsen-Wittstock factorization**)
Let $E \subset B(H)$ *and* $F \subset B(K)$ *be two operator spaces. Let* $u : E \to F$ *be a c.b. map. Then there are a Hilbert space* $\widetilde{H}$*, a representation* $\pi : B(H) \to B(\widetilde{H})$ *and two bounded operators* $a, b \in B(K, \widetilde{H})$ *such that*

$$u(x) = L_{b^*}\, \pi(x) R_a, \quad \forall\, x \in E.$$

Namely, $u = L_{b^*} \circ R_a \circ \pi\big|_E$*. Moreover,*

$$\|u\|_{cb} = \inf \big\{ \|a\|\, \|b\| \big\},$$

where the infimum is taken over all factorizations of u *as above.*

Corollary 2.5 (**Hahn-Banach type extension**)
Every c.b. map $u : E \to B(K)$ *admits a c.b. extension* $\tilde{u} : B(H) \to B(K)$ *such that* $\|\tilde{u}\|_{cb} = \|u\|_{cb}$*.*

Proof. Take a factorization $u = L_{b^*} \circ R_a \circ \pi\big|_E$ such that $\|u\|_{cb} = \|a\|\, \|b\|$. Then $\tilde{u} = L_{b^*} \circ R_a \circ \pi$ is the desired extension of u. ☐

Let $E \subset B(H)$ be an operator space. Then E inherits the order of $B(H)$, which allows us to define positive and completely positive maps on E. Thus a map $u : E \to B(K)$ is c.p. if u_n sends the positive part of $\mathbb{M}_n(E)$ into that of $\mathbb{M}_n(B(K))$ for every n.

Corollary 2.6 (**Decomposability of c.b. maps**)
Every c.b. map $u : E \to B(K)$ *is decomposable in the sense that there are four c.p. maps* u_k *such that*

$$u = u_1 - u_2 + i(u_3 - u_4)$$

with $\|u_k\|_{cb} \le \|u\|_{cb}$*.*

Proof. Let $u = L_{b^*} \circ R_a \circ \pi\big|_E$ according to Theorem 2.4. Then the corollary immediately follows from the polarization identity with

$$u_k = \frac{1}{4}\, C_{a+i^k b} \circ \pi\big|_E\,, \quad 1 \le k \le 4,$$

where C_a is the multiplication by a from the right and by a^* from the left. ☐

Usually, we do not distinguish completely isometric operator spaces. The distance between two operator spaces is measured by the operator space analogue of the Banach-Mazur distance in the theory of Banach spaces. The Banach-Mazur distance of two Banach spaces E and F is

$$d(E,F) = \inf \big\{ \|u^{-1}\|\, \|u\| \ :\ u : E \to F \text{ is an isomorphism} \big\}.$$

If E and F are isomorphic, $d(E,F) < \infty$; otherwise, $d(E,F) = \infty$.

Definition 2.7 Given two operator spaces E and F define

$$d_{cb}(E,F) = \inf \big\{ \|u^{-1}\|_{cb}\, \|u\|_{cb} \ :\ u : E \to F \text{ is a complete isomorphism} \big\}.$$

Pisier [P1] proved that if $\dim E = \dim F = n$, then $d_{cb}(E,F) \le n$. We will see that the upper bound is attained for the pair (C^n, R^n). To this end, let us introduce a general definition.

Definition 2.8 An operator space E is called *homogeneous* if every bounded map on E is c.b. and $\|u\|_{cb} = \|u\|$. E is called *Hilbertian* if E is isometric to a Hilbert space.

It is easy to see that $\min(E)$ and $\max(E)$ are homogeneous. On the other hand, C and R are homogeneous Hilbertian operator spaces. We refer to Chapiter 10 of [P3] for the proofs of the following two results.

Proposition 2.9 *Let E and F be two n-dimensional Hilbertian homogeneous operator spaces. Let $(e_1, ..., e_n)$ and $(f_1, ..., f_n)$ be orthonormal bases of E and F, respectively. Let $u : E \to F$ be the map defined by $u(e_i) = f_i$. Then*

$$d_{cb}(E, F) = \|u^{-1}\|_{cb} \|u\|_{cb} \,.$$

Corollary 2.10 *$d_{cb}(C^n, R^n) = n$ and $d_{cb}(C^n, \min(\ell_2^n))) = \sqrt{n}$. Consequently, C, R and $\min(\ell_2)$ are not completely isomorphic each other.*

Exercices:

1) Let E be a Banach space and F an operator space. Prove that any bounded maps $u : F \to \min(E)$ and $v : \max(E) \to F$ are c.b. and $\|u\|_{cb} = \|u\|$, $\|v\|_{cb} = \|v\|$.

2) Prove that C and R are homogeneous.

3) Prove Corollary 2.10.

4) Let $u : C \to R$. Prove $\|u\|_{cb} = \|u\|_{HS}$, where $\|u\|_{HS}$ denotes the Hilbert-Schmidt norm of u regarded as an operator on ℓ_2, i.e.,

$$\|u\|_{HS} = \big(\sum_{i=1}^{\infty} \|u(e_i)\|_2^2\big)^{1/2} .$$

3 Ruan's theorem: abstract operator spaces

The definition of concrete operator spaces presented in the previous section has a major drawback: it does not allow to do basic operations on concrete operator spaces. For instance, it is not clear at all how to introduce a nice matricial structure on the Banach dual E^* of E which reflects the one of E. This drawback is resorbed in Ruan's definition of abstract operator spaces.

We have already seen that a concrete operator space $E \subset B(H)$ possesses a natural matricial structure inherited from that of $B(H)$: for each n, $\mathbb{M}_n(E) \subset B(\ell_2^n(H))$ is again an operator space, equipped with the norm $\|\cdot\|_n$ induced by that of $B(\ell_2^n(H))$. This sequence $(\|\cdot\|_n)$ of matricial norms clearly satisfy the following properties

- (R$_1$): $\|\alpha x \beta\|_n \le \|\alpha\| \, \|x\|_n \|\beta\|, \ \forall\, \alpha, \beta \in \mathbb{M}_n, x \in \mathbb{M}_n(E), n \ge 1.$
- (R$_2$): $\|x \oplus y\|_{n+m} \le \max\big(\|x\|_n, \ \|y\|_m\big), \forall\, x \in \mathbb{M}_n(E), y \in \mathbb{M}_m(E), n, m \ge 1.$

Here the product is the usual matrix product. $x \oplus y$ denotes the $(n+m) \times (n+m)$-matrix

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Theorem 3.1 (Ruan's characterization)
Let E be a vector space. Assume that each $\mathbb{M}_n(E)$ is equipped with a norm $\|\cdot\|_n$. If these norms $\|\cdot\|_n$ satisfy Ruan's axioms (R_1) and (R_2), then there are a Hilbert space H and a linear map $J : E \to B(H)$ such that

$$J_n = \mathrm{id}_{\mathbb{M}_n} \otimes J : \mathbb{M}_n(E) \to \mathbb{M}_n(B(H)) \quad \textit{is isometric for every}\, n.$$

In other words, the sequence $(\|\cdot\|_n)$ comes from the operator space structure of E given by the embedding $J : E \to B(H)$.

This theorem is proved in [R] (see also [ER] for an alternate proof).

Definition 3.2 An (*abstract*) *operator space* is a Banach space E together with a sequence $(\|\cdot\|_n)$ of norms satisfying (R_1) and (R_2) (with $\|\cdot\|_1$ equal to the original norm of E).

Henceforth, we will drop the adjective "concrete" or "abstract" by saying only operator spaces. Thus to have an operator space structure on a Banach space E is to have a sequence of matricial norms verifying Ruan's axioms. In the remainder of this section we present some basic operations on operator spaces. The complex interpolation is postponed, however, to the next one, where Pisier's operator Hilbert space will be also introduced.

Spaces of c.b. maps. If E and F are two operator spaces, $CB(E,F)$ denotes again the space of all c.b. maps from E to F. This is a Banach space equipped with the norm $\|\cdot\|_{cb}$. Now we wish to turn $CB(E,F)$ an operator space. Let $u=(u_{ij})\in\mathbb{M}_n(CB(E,F))$. We view u as a map from E into $\mathbb{M}_n(F)$ by defining $u(x)=\big(u_{ij}(x)\big)$ for $x\in E$. Then the matricial norm on $\mathbb{M}_n(CB(E,F))$ is defined by

$$\|u\|_n=\big\|u:E\to\mathbb{M}_n(F)\big\|_{cb}\,.$$

Namely, we have the identification

$$\mathbb{M}_n(CB(E,F))=CB(E,\mathbb{M}_n(F)).$$

It is easy to show that Ruan's axioms are verified. Thus $CB(E,F)$ becomes an operator space.

Duality. Specializing the previous discussion to $F=\mathbb{C}$, we see that $CB(E,\mathbb{C})$ is an operator space. However, Proposition 2.3 implies that $CB(E,\mathbb{C})=E^*$ isometrically. Therefore, E^* becomes an operator space. The norm of $\mathbb{M}_n(E^*)$ is that of $CB(E,\mathbb{M}_n)$. This is usually called the *standard dual* of E. We will simply say the dual of E since only standard duals are used in the sequel. The bidual $E^{**}=(E^*)^*$ is an operator space too. Then it is easy to check that the natural inclusion $E\hookrightarrow E^{**}$ is completely isometric. This allows to view E as a subspace of E^{**}.

Thus the duals of C*-algebras and the preduals of von Neumann algebras are operator spaces.

Let $u:E\to F$ be a map between two operator spaces. Then u is c.b. iff its adjoint $u^*:F^*\to E^*$ is c.b.. If this is the case, $\|u\|_{cb}=\|u^*\|_{cb}$.

Quotient. Let E be an operator space and $F\subset E$ a closed subspace. We equip $\mathbb{M}_n(E/F)$ with the quotient norm of $\mathbb{M}_n(E)/\mathbb{M}_n(F)$. Then it is easy to check that these norms satisfy (R_1) and (R_2), so E/F becomes an operator space. The usual duality between subspaces and quotients in Banach space theory remains available now:

$$F^*=\frac{E^*}{F^\perp}\quad\text{and}\quad\big(\frac{E}{F}\big)^*=F^\perp\quad\text{completely isometrically.}$$

Direct sum. Let (E_k) be a sequence of operator spaces, $E_k\subset B(H_k)$ (the sequence may be finite). Let $\ell_\infty((E_k))$ denote the space of all sequences (x_k) with $x_k\in E_k$ such that $\sup_k\|x_k\|<\infty$. This is a Banach space when equipped with the norm

$$\|(x_k)\|=\sup_k\|x_k\|.$$

$\ell_\infty((E_k))$ naturally inherits a matricial structure from $\ell_\infty((B(H_k)))$, the latter being a C*-algebra. Thus we have

$$\mathbb{M}_n\big(\ell_\infty((E_k))\big)=\ell_\infty\big((\mathbb{M}_n(E_k))\big).$$

$c_0((E_k))$ denotes the subspace of $\ell_\infty((E_k))$ consisting of all (x_k) such that $\|x_k\|\to 0$.

On the other hand, we define $\ell_1((E_k))$ to be the space of all sequences (x_k) with $x_k\in E_k$ such that $\sum_k\|x_k\|<\infty$. This is again a Banach space with the natural norm. Recall that

$$\big(\ell_1((E_k))\big)^*=\ell_\infty((E_k^*))\quad\text{isometrically.}$$

Thus $\ell_1((E_k))$ is a predual of $\ell_\infty((E_k))$, which allows us to view $\ell_1((E_k))$ as an operator space too. We often use the notations

$$\bigoplus_k {}_\infty\, E_k \quad \text{and} \quad \bigoplus_k {}_1\, E_k$$

instead of $\ell_\infty((E_k)_k)$ and $\ell_1((E_k)_k)$, respectively. If all E_k are equal, these spaces are denoted by $\ell_\infty(E)$ and $\ell_1(E)$, respectively. In particular, if $E = \mathbb{C}$, we recover ℓ_∞ and ℓ_1.

Let us introduce the continuous version of $c_0(E)$. Let Ω be a locally compact topological space and $E \subset B(H)$ an operator space. Let $C_0(\Omega; E)$ denote the space of continuous functions from Ω to E which vanish at infinity. $C_0(\Omega; E)$ is equipped with the uniform norm

$$\|f\| = \sup_{\omega\in\Omega} \|f(\omega)\|.$$

Then $C_0(\Omega; E) \subset C_0(\Omega; B(H))$. The latter space is a C*-algebra. In this way, we turn $C_0(\Omega; E)$ an operator space.

Sum and intersection. Let (E_0, E_1) be a couple of operator spaces. Assume (E_0, E_1) is *compatible* in the sense that both E_0 and E_1 continuously embed into a common topological vector space V. This allows us to define

$$E_0 \cap E_1 = \big\{x \in V \ : \ x \in E_0,\ x \in E_1\big\}$$

and

$$E_0 + E_1 = \big\{x \in V \ : \ \exists\, x_0 \in E_0,\ x_1 \in E_1 \text{ s.t. } x = x_0 + x_1\big\},$$

equipped respectively with the intersection and sum norms

$$\|x\|_{E_0\cap E_1} = \max\big(\|x\|_{E_0}\,,\ \|x\|_{E_1}\big),$$

$$\|x\|_{E_0+E_1} = \inf\big\{\|x_0\|_{E_0} + \|x_1\|_{E_1} \ : \ x = x_0 + x_1, x_0 \in E_0,\ x_1 \in E_1\big\}.$$

Note that $E_0 \cap E_1$ can be regarded as the diagonal subspace of $E_0 \oplus_\infty E_1$. On the other hand, $E_0 + E_1$ is identifiable with the quotient space of $E_0 \oplus_1 E_1$ by Δ, where $\Delta = \{(x_0, x_1) \ : \ x_0 + x_1 = 0\}$. Equipped with the operator space structures induced by those of $E_0 \oplus_\infty E_1$ and $E_0 \oplus_1 E_1$, respectively, $E \cap E_1$ and $E_0 + E_1$ become operator spaces too.

Let us consider an important example. Take the column and row spaces C and R and view them as compatible by identifying both of them with ℓ_2 at the Banach space level. Thus if $x \in C$, we write

$$x = \sum_i x_i e_{i1} \quad \text{with} \quad x_i \in \mathbb{C}.$$

Recall that

$$\|x\|_C = (\sum_i |x_i|^2)^{1/2} = \|(x_i)\|_{\ell_2} = \|\sum_i x_i e_{1i}\|_R\,.$$

Then the compatibility above means that x is identified with the sequence (x_i). Accordingly, we often identify the canonical bases (e_{i1}) of C and (e_{1i}) of R with (e_i) of ℓ_2. Thus as Banach spaces, $C \cap R = C = R (= \ell_2)$. But this is no longer true in the category of operator spaces.

Let $x \in \mathbb{M}_n(C \cap R) = \mathbb{M}_n(C) \cap \mathbb{M}_n(R)$. Write

$$x = \sum_i x_i \otimes e_i \quad \text{with} \quad x_i \in \mathbb{M}_n.$$

Then

$$\begin{aligned} \|x\|_{\mathbb{M}_n(C\cap R)} &= \max\big(\|x\|_{\mathbb{M}_n(C)}\,,\ \|x\|_{\mathbb{M}_n(R)}\big) \\ &= \max\big(\|\sum_i x_i^* x_i\|^{1/2}\,,\ \|\sum_i x_i x_i^*\|^{1/2}\big). \end{aligned}$$

Using this, we easily check that $C \cap R$ is not completely isomorphic to C. Indeed, using Proposition 2.3, one easily shows that their n-dimensional versions satisfy the following

$$d_{cb}(C^n,\ C^n \cap R^n) = \sqrt{n}.$$

The operator space structure of $C + R$ is a little bit more complicated. Using the complete isomorphism between $E_1 \oplus_\infty E_2$ and $E_1 \oplus_1 E_2$ (see Exercice 6 below), we deduce that for $x = \sum_i x_i \otimes e_i \in \mathbb{M}_n(C + R)$

$$\frac{1}{2}\, \|x\|_{\mathbb{M}_n(C+R)} \le \inf\big\{ \big\| \sum_i y_i^* y_i \big\|^{1/2} + \big\| \sum_i z_i z_i^* \big\|^{1/2} \big\} \le 2\|x\|_{\mathbb{M}_n(C+R)}\,,$$

where the infimum runs over all decompositions $x_i = y_i + z_i$ with $y_i,\, z_i \in \mathbb{M}_n$. We will give later a complete isometric description of $C + R$ in terms of the trace class S_1.

Exercices:

1) Prove that $C^* \simeq R$ and $R^* \simeq C$ completely isometrically. More precisely, the map $\xi \in C^* \mapsto \sum_i \xi(e_{i1}) e_{1i}$ establishes a complete isometry between C^* and R.

2) Show that the natural inclusion $E \hookrightarrow E^{**}$ is completely isometric.

3) Let $F \subset E$ be operator spaces. Prove

$$F^* = \frac{E^*}{F^\perp} \quad \text{and} \quad \big(\frac{E}{F}\big)^* = F^\perp \quad \text{completely isometrically.}$$

4) Prove that a map $u : E \to F$ is c.b. iff its adjoint $u^* : F^* \to E^*$ is c.b.. Moreover, $\|u\|_{cb} = \|u^*\|_{cb}$.

5) Prove

$$\big(c_0((E_k))\big)^* = \ell_1((E_k^*)) \quad \text{and} \quad \big(\ell_1((E_k))\big)^* = \ell_\infty((E_k^*)) \quad \text{completely isometrically.}$$

6) Prove that the natural inclusion from $\ell_1((E_k))$ into $\ell_\infty((E_k))$ is completely contractive. Conversely, prove that the c.b. norm of the formal identity from $E_1 \oplus_\infty \cdots \oplus_\infty E_n$ to $E_1 \oplus_1 \cdots \oplus_1 E_n$ is equal to n.

7) Let (E_0, E_1) be a compatible couple of operator spaces such that $E_0 \cap E_1$ is dense in both E_0 and E_1. Prove

$$\big(E_0 \cap E_1\big)^* = E_0^* + E_1^*, \quad \big(E_0 + E_1\big)^* = E_0^* \cap E_1^* \quad \text{completely isometrically.}$$

4 Complex interpolation and operator Hilbert spaces

Hilbert spaces are in the center of the family of Banach spaces and play a crucial role there. It is thus natural to find their operator space analogues. One way to define operator Hilbert spaces is by complex interpolation, which is of great importance for its own right in operator space theory. Our references for this section are [P1] and [P3].

Interpolation. We first recall the definition of complex interpolation for Banach spaces. Let (E_0, E_1) be a compatible couple of complex Banach spaces. Let $S = \{z \in \mathbb{C} \;:\; 0 \le \mathrm{Re}\, z \le 1\}$, a strip in $\mathbb{C}$. Let $\mathcal{F}(E_0, E_1)$ be the family of all functions $f :\ S \to E_0 + E_1$ satisfying the following conditions:

- f is continuous on S and analytic in the interior of S;

- $f(k+it) \in E_k$ for all $t \in \mathbb{R}$ and the function $t \mapsto f(k+it)$ is continuous from $\mathbb{R}$ to E_k for $k=0$ and $k=1$;
- $\lim_{|t|\to\infty} \|f(k+it)\|_{E_k} = 0$ for $k=0$ and $k=1$.

We equip $\mathcal{F}(E_0, E_1)$ with the norm:

$$\big\|f\big\|_{\mathcal{F}(E_0,E_1)} = \max\big\{\sup_{t\in\mathbb{R}} \big\|f(it)\big\|_{E_0}\,,\ \sup_{t\in\mathbb{R}} \big\|f(1+it)\big\|_{E_1}\big\}.$$

Then it is a routine exercise to check that $\mathcal{F}(E_0, E_1)$ is a Banach space. For $0<\theta<1$ the complex interpolation space $E_\theta = (E_0,\ E_1)_\theta$ is defined as the space of all those $x \in E_0 + E_1$ for which there exists $f \in \mathcal{F}(E_0, E_1)$ such that $f(\theta) = x$. Equipped with

$$\|x\|_\theta = \inf\big\{\big\|f\big\|_{\mathcal{F}(E_0,E_1)}\ :\ f(\theta)=x,\ f\in\mathcal{F}(E_0,E_1)\big\},$$

E_θ becomes a Banach space. Note that by the maximum principle, the map $f \mapsto f(\theta)$ is a contraction from $\mathcal{F}(E_0, E_1)$ to $E_0 + E_1$. Then $(E_0,\ E_1)_\theta$ can be isometrically identified with the quotient of $\mathcal{F}(E_0, E_1)$ by the kernel of this map.

Now assume E_0 and E_1 are operator spaces. Then $(\mathbb{M}_n(E_0), \mathbb{M}_n(E_1))$ is again compatible for any $n \geq 1$. This allows to define

$$\mathbb{M}_n(E_\theta) = \big(\mathbb{M}_n(E_0),\ \mathbb{M}_n(E_1)\big)_\theta\,.$$

It is easy to show that Ruan's axioms are satisfied, so E_θ is an operator space. Let us express E_θ as a quotient of a subspace of $C_0(\mathbb{R}; E_0) \oplus_\infty C_0(\mathbb{R}; E_1)$. Using Poisson integral, one sees that $C_0(\mathbb{R}; E_0)\oplus_\infty C_0(\mathbb{R}; E_1)$ is just the space of functions $f : S \to E_0+E_1$ satisfying the same conditions as above for $\mathcal{F}(E_0, E_1)$ but only with "analytic" replaced by "harmonic". Then $\mathcal{F}(E_0, E_1)$ is the subspace of $C_0(\mathbb{R}; E_0) \oplus_\infty C_0(\mathbb{R}; E_1)$ consisting of all analytic functions. Therefore, E_θ is the quotient space of $\mathcal{F}(E_0, E_1)$ by the subspace of all f such that $f(\theta) = 0$.

Operator Hilbert spaces. Let H be a complex Hilbert space. Fixing an orthonormal basis $(e_i)_{i\in I}$ in H, we can identify H with $\ell_2(I)$. The classical Riesz representation theorem asserts that H^* is isometric to the conjugate $\bar{H}$ of H. The latter space is H itself with the same norm but with conjugate multiplication: $\lambda \cdot x = \bar{\lambda} x$ for $\lambda \in \mathbb{C}$ and $x \in H$. Viewed as a vector in $\bar{H}$, a vector $x \in H$ is often denoted by $\bar{x}$. Thus if $x = (x_i) \in \ell_2(I)$, then $\bar{x} = (\bar{x}_i)_{i\in I}$. The map $x \mapsto \bar{x}$ establishes an anti-linear isometry between H and $\bar{H}$. Consequently, $\overline{H^*}$ is isometric to H. In fact, the conjugation can be defined for any Banach space X. It is a well-known elementary fact that the Hilbert spaces are only Banach spaces X such that $\overline{X^*}$ is isometric to X.

Now we wish to consider the operator space analogue of this property. The resulting spaces are the so-called operator Hilbert spaces, introduced by Pisier. Note that if E is an operator space, so is $\bar{E}$ by defining $\mathbb{M}_n(\bar{E}) = \overline{\mathbb{M}_n(E)}$.

Theorem 4.1 *Let I be an index set. For any $x = \sum_i x_i \otimes e_i \in \mathbb{M}_n(\ell_2(I))$ define*

$$\|x\|_n = \big\|\sum_i x_i \otimes \bar{x}_i\big\|^{1/2}_{\mathbb{M}_n\otimes\overline{\mathbb{M}}_n} = \big\|\sum_i x_i \otimes \bar{x}_i\big\|^{1/2}_{\mathbb{M}_{n^2}}\,.$$

Then $(\|\cdot\|_n)$ satisfy Ruan's axioms. The resulting operator space is denoted by $OH(I)$. Moreover, $OH(I)$ is the unique, up to complete isometry, operator space structure on $\ell_2(I)$ such that $\overline{OH(I)^*}$ is completely isometric to $OH(I)$.

If $I = \mathbb{N}$ or $I = \{1, ..., n\}$, we denote $OH(I)$ simply by OH of OH^n. The space OH can be also obtained by interpolating the column and row spaces.

Theorem 4.2 *$OH = (C,\ R)_{\frac{1}{2}}$ completely isometrically.*

Exercices:

1) Let (E_0, E_1) and (F_0, F_1) be two compatible couples of operator spaces. Let $T : E_0 + E_1 \to F_0 + F_1$ be a linear map such that $T\big|_{E_k} : E_k \to F_k$ is c.b. of cb-norm c_k for $k = 0, 1$. Then T is c.b. from E_θ to F_θ for any $0 < \theta < 1$ and of cb-norm $\le c_0^{1-\theta} c_1^{\theta}$.

2) Show that OH is homogeneous and $d_{cb}(C^n,\ OH^n) = \sqrt{n}$. Consequently, OH is not completely isomorphic to C.

5 Vector-valued noncommutative L_p-spaces

Since operator spaces are quantized Banach spaces, noncommutative L_p-spaces are quantized L_p-spaces. Thus it is not surprising that noncommutative L_p-spaces play the role in operator space theory as the usual L_p-spaces do in the category of Banach spaces. In this section we introduce the natural operator space structures on noncommutative L_p and Pisier's vector-valued Schatten classes. We will need essentially the cases $p = \infty$ and $p = 1$ (so the case $1 < p < \infty$ can be skipped). For these special cases what is presented below becomes extremely simple for noncommutative L_∞ and L_1 are nothing but von Neumann algebras and their preduals.

5.1 Noncommutative L_p-spaces. Let M be a von Neumann algebra equipped with a normal faithful tracial state τ. For $1 \le p < \infty$ and $x \in M$ define

$$\|x\|_p = [\tau(|x|^p)]^{1/p}, \quad \text{where} \quad |x| = (x^*x)^{1/2}\,.$$

Then $(M, \|\cdot\|_p)$ is a normed space, whose completion is the noncommutative L_p-space associated with (M, τ), denoted by $L_p(M)$. By convention, $L_\infty(M) = M$ with the operator norm. Then for any $1 \le p < \infty$, the dual space of $L_p(M)$ is $L_{p'}(M)$ (p' being the conjugate index of p):

$$L_p(M)^* = L_{p'}(M) \quad \text{isometrically}$$

with respect to the duality bracket

$$\langle x,\ y\rangle = \tau(xy), \quad x \in L_p(M),\ y \in L_{p'}(M).$$

Consequently, $L_1(M)$ is the predual of M.

Let us consider two special cases. The first is where M is commutative, say, $M = L_\infty(\Omega, \mu)$ for some probability space (Ω, μ) (μ can be, of course, assumed to be a σ-finite measure). Then we recover the usual L_p-spaces $L_p(\Omega)$. The second case concerns $M = B(\ell_2^n)$, equipped with the usual trace Tr on $B(\ell_2^n)$ (which can be normalized to a state if we wish). Then we get the Schatten classes S_p^n. The infinite dimensional algebra $B(\ell_2)$ is not covered by this definition since the usual trace Tr on $B(\ell_2)$ is not finite. However, it still has a nice trace in the sense that it is *normal, semifinite and faithful.* The preceding construction can be done for normal semifinite faithful traces too. The resulting spaces for $(B(\ell_2), \mathrm{Tr})$ are the Schatten classes S_p. Recall that S_1 and S_2 are respectively the trace and Hilbert-Schmidt classes. Note that by definition S_∞ is the whole $B(\ell_2)$.

The previous definition does not apply to type III von Neumann algebras. There are several equivalent constructions of noncommutative L_p-spaces in the type III case. Here we adopt the one by Kosaki [Ko] via complex interpolation. Let M be a von Neumann algebra equipped with a normal faithful state φ. Define $L_\infty(M) = M$ as before and $L_1(M) = M_*$. Consider the left injection j of $L_\infty(M)$ into $L_1(M)$ by $j(x) = x\varphi$. The faithfulness of φ implies that j is injective and its range is dense in $L_1(M)$. This injection makes $(L_\infty(M),\ L_1(M))$ compatible. Thus we can consider the complex interpolation spaces between them. Now for $1 < p < \infty$ define

$$L_p(M) = \big(L_\infty(M),\ L_1(M)\big)_{\frac{1}{p}}\,.$$

We refer to the survey paper [PX] for more information and historical references on noncommutative L_p-spaces.

5.2 Operator space structures on noncommutative L_p. Now we turn to describe the natural operator space structure on $L_p(M)$ (see [P3] for more information). For $p = \infty$, $L_\infty(M) = M$ has its natural operator space structure as a von Neumann algebra. This also yields an operator space structure on M^*, the standard dual of M. To deal with the case $p = 1$ we consider the opposite von Neumann algebra $M^{\rm op}$ of M. $M^{\rm op}$ is the same as M but with the new multiplication which is opposite to that of M: $x \cdot y$ in $M^{\rm op}$ is equal to $yx \in M$. Note that if M acts on H, then $M^{\rm op}$ acts on H^* and coincides with $\{x^t \ : \ x \in M\}$, where x^t denotes the transpose (= Banach space adjoint) of x. It is clear that the map $x \mapsto \overline{x^*}$ establishes an isomorphism between M and $\overline{M^{\rm op}}$. Thus $L_1(M)$ is isometric to $L_1(M^{\rm op})$ at the Banach space level. This allows us to equip $L_1(M)$ with the operator space structure inherited from $(M^{\rm op})^*$. The main reason for this choice is that it insures that the equality $L_1(\mathbb{M}_n \otimes M) = S_1^n \widehat{\otimes} L_1(M)$ (operator space projective tensor product) holds true. Finally, the operator space structure of $L_p(M)$ is obtained by complex interpolation. It is worth to mention that $L_2(M)$ is an operator Hilbert space by virtue of a theorem of Pisier.

Thus for every σ-finite measure space (Ω, μ), the commutative L_p-spaces $L_p(\Omega)$ are equipped with their natural operator space structures. In particular, the ℓ_p are operator spaces and $\ell_2 = OH$.

The same happens to the Schatten classes S_p too. If $1 \le p < \infty$, the dual space of S_p is $S_{p'}$ with respect to the so-called parallel duality bracket

$$\langle x,\ y\rangle = \mathrm{Tr}(xy^t) = \sum_{i,j} x_{ij} y_{ij}$$

for $x = (x_{ij}) \in S_p$ and $y = (y_{ij}) \in S_{p'}$. This is due to our definition that S_1 is defined as the predual of $B(\ell_2)^{\rm op}$.

5.3 Vector-valued Schatten classes. We wish to define Schatten classes with values in operator spaces. To this end we first recall the *minimal tensor product* in the category of operator spaces. Let $E \subset B(H)$ and $F \subset B(K)$ be two operator spaces. Let $x \in B(H)$ and $y \in B(K)$. The tensor $x \otimes y$ is an operator on the Hilbert space tensor product $H \otimes K$ defined by $x \otimes y(\xi \otimes \eta) = x(\xi) \otimes y(\eta)$. It is easy to check that $x \otimes y$ is bounded and $\|x \otimes y\| = \|x\|\,\|y\|$. Consequently, the algebraic tensor product $E \otimes F$ is a vector subspace of $B(H \otimes K)$. Then the minimal tensor product $E \otimes_{\min} F$ is defined to be the closure of $E \otimes F$ in $B(H \otimes K)$.

Now let E be an operator space. Define $S_\infty[E]$ to be $S_\infty \otimes_{\min} E$. The elements in $S_\infty[E]$ are often represented as infinite matrices with entries in E.

To define $S_1[E]$ we use duality. The operator space structure to be put in $S_1[E]$ will be such that the dual space of $S_1[E]$ is $S_\infty[E^*]$. More precisely, let $u \in S_1 \otimes E$. Write

$$u = \sum_k a_k \otimes x_k \quad \text{with} \quad a_k \in S_1,\ x_k \in E.$$

Consider u as a linear functional on $S_\infty[E^*]$ as follows. For $v = \sum_j b_j \otimes \xi_j \in S_\infty \otimes E^*$ define

$$\langle u, v\rangle = \sum_{k,j} \langle a_k, b_j\rangle \langle x_k, \xi_j\rangle = \sum_{k,j} \mathrm{Tr}(a_k b_j^t)\, \xi_j(x_k).$$

Then the norm of u is defined to be the linear functional norm of u on $S_\infty[E^*]$, which coincides with the norm of u in $CB(S_\infty[E^*], \mathbb{C})$ (see Proposition 2.3). We define $S_1[E]$ to be the closure of $S_1 \otimes E$ with respect to this norm. Next, we have to introduce a norm $\|\cdot\|_n$ on $\mathbb{M}_n(S_1[E])$ for any n. This is now easy. As before, every $u \in \mathbb{M}_n(S_1 \otimes E)$ induces a linear map from $S_\infty[E^*]$ to $\mathbb{M}_n$. Then define

$$\|u\|_n = \|u\|_{CB(S_\infty[E^*], \mathbb{M}_n)}.$$

It is then routine to check that these norms satisfy Ruan's axioms. Therefore, $S_1[E]$ becomes an operator space.

Having defined $S_\infty[E]$ and $S_1[E]$, we define $S_p[E]$ by interpolation for any $1 < p < \infty$:

$$S_p[E] = \big(S_\infty[E],\ S_1[E]\big)_{1/p}\,.$$

The elements of $S_p[E]$ are often represented as infinite matrices with entries in E. It is not hard to show that finite matrices (i.e., those with only a finite number of nonzero entries) are dense in $S_p[E]$ for $p < \infty$.

The following theorem of Pisier is very useful. The reader is referred to [P2] for its proof.

Theorem 5.1 *Let $1 \le p < \infty$.*

i) *Any $x = (x_{ij}) \in S_p[E]$ admits a factorization $x = ayb$ with $a, b \in S_{2p}$ and $y \in S_\infty[E]$. Here the product is the usual matrix product. Moreover, we have*

$$\|x\|_{S_p[E]} = \inf_{x=ayb} \big\{\|a\|_{2p}\, \|y\|_{S_\infty[E]}\, \|b\|_{2p}\big\}.$$

ii) *Conversely, for any $x = (x_{ij}) \in S_\infty[E]$*

$$\|x\|_{S_\infty[E]} = \sup\big\{\|ayb\|_{S_p[E]}\ :\ a, b \in S_{2p},\ \|a\|_{2p} \le 1, \|b\|_{2p} \le 1\big\}.$$

Corollary 5.2 *Let E and F be two operator spaces. Let $1 \le p < \infty$. Then a linear map $u : E \to F$ is c.b. iff*

$$\sup_n \big\|I_{S_p^n} \otimes u : S_p^n[E] \to S_p^n[F]\big\| < \infty;$$

moreover in this case the supremum above is equal to $\|u\|_{cb}$. Alternatively, u is c.b. iff $I_{S_p} \otimes u$ extends to a bounded map from $S_p[E]$ to $S_p[F]$.

Proof. Assume that u is c.b., i.e.,

$$\|u\|_{cb} = \sup_n \big\|I_{S_\infty^n} \otimes u : S_\infty^n[E] \to S_\infty^n[F]\big\| < \infty.$$

Let $x = (x_{ij})$ be a matrix in $S_p^n[E]$ with norm less than 1. Then x admits a factorization $x = ayb$ with

$$\|a\|_{S_{2p}^n} \le 1, \quad \|y\|_{S_\infty^n[E]} \le 1, \quad \|b\|_{S_{2p}^n} \le 1.$$

We have

$$I \otimes u(x) = a(I \otimes u)(y)b.$$

Therefore,

$$\big\|I \otimes u(x)\big\|_{S_p^n[F]} \le \|a\|_{S_{2p}^n}\, \big\|I \otimes u(y)\big\|_{S_\infty^n[F]}\, \|b\|_{S_{2p}^n} \le \|u\|_{cb}\,;$$

whence

$$\sup_n \big\|I_{S_p^n} \otimes u : S_p^n[E] \to S_p^n[F]\big\| \le \|u\|_{cb}\,.$$

The converse is proved similarly. □

The previous corollary is very useful notably for subspaces of noncommutative L_p-spaces for the following reason. Given $E \subset L_p(M)$ it is usually difficult to determine the norm of $\mathbb{M}_n(E)$; but it is extremely simple to describe the norm of $S_p[E]$, as shows the next paragraph.

We will only need the case where E is a subspace of a noncommutative $L_p(M)$ in which the previous theory becomes much simpler. Note that there is a natural algebraic identification of $L_p(\mathbb{M}_n \otimes M)$ with $\mathbb{M}_n(L_p(M))$. Then $S_p^n[L_p(M)]$ is nothing but the linear space $\mathbb{M}_n(L_p(M))$

equipped with the norm of $L_p(\mathbb{M}_n \otimes M)$. More generally, if $E \subset L_p(M)$ is a closed subspace, the norm of $S_p^n[E]$ is induced by that of $S_p^n[L_p(M)]$. In the infinite dimensional case, $S_p[L_p(M)]$ is completely isometrically identified with $L_p(B(\ell_2)\bar{\otimes}M)$ for all $1 \le p < \infty$. If $E \subset L_p(M)$, then $S_p[E]$ is the closure of $S_p \otimes E$ in $L_p(B(\ell_2)\bar{\otimes}M)$.

5.4 Column and row p-spaces. The column and row spaces, C and R, are pillars of the whole theory of operator spaces. Recall that C and R are respectively the (first) column and row subspaces of S_∞. By analogy, let C_p (resp. R_p) denote the first column (resp. row) subspace of S_p. Since S_2 is an OH space, $C_2 \simeq R_2 \simeq OH$. The n-dimensional versions of these spaces are denoted by C_p^n and R_p^n.

Now let E be an operator space. We denote by $C_p[E]$ (resp. $R_p[E]$) the closure of $C_p \otimes E$ (resp. $R_p \otimes E$) in $S_p[E]$. If E is a subspace of a noncommutative $L_p(M)$, the norm of $C_p[E]$ is easy to determine. We consider only the case where M is semifinite. For any finite sequence $(x_k) \subset E$

$$\big\|\sum_k x_k \otimes e_k\big\|_{C_p[E]} = \big\|\big(\sum_k x_k^* x_k\big)^{1/2}\big\|_{L_p(M)}\ ,$$

where (e_k) denotes the canonical basis of C_p. More generally, if $a_k \in C_p$, then

$$\big\|\sum_k x_k \otimes a_k\big\|_{C_p[E]} = \big\|\big(\sum_k \langle a_k, a_j\rangle x_k^* x_j\big)^{1/2}\big\|_{L_p(M)}\ ,$$

where $\langle\ ,\ \rangle$ denotes the scalar product in C_p. (In terms of matrix product, $\langle a_k, a_j\rangle = a_k^* a_j$.) We also have a similar description for $R_p[E]$.

We end this section by describing the operator space structure of $C + R$ with help of Corollary 5.2. To this end we use the identification that $C \simeq R_1$ and $R \simeq C_1$ (see Exercice 2 below). Thus $C+R \simeq R_1 + C_1$, which is the quotient of $R_1 \oplus_1 C_1$ by the subspace $\{(x, y)\,;\ x+y = 0\}$. Therefore, for any finite sequence $(x_k) \subset S_1$

$$\begin{aligned} \big\|(x_k)\big\|_{S_1[C+R]} &= \inf\big\{\big\|(y_k)\big\|_{S_1[R_1]} + \big\|(z_k)\big\|_{S_1[C_1]}\big\} \\ &= \inf\big\{\big\|\big(\sum_k (y_k y_k^*)^{1/2}\big)\big\|_{S_1} + \big\|\big(\sum_k (z_k^* z_k)^{1/2}\big)\big\|_{S_1}\big\}, \end{aligned}$$

where the infimum runs over all decompositions of $x_k = y_k + z_k$ in S_1. It follows that $S_1[C+R] = C_1[S_1] + R_1[S_1]$ with equal norms (even completely isometrically).

Exercices:

1) Let ℓ_1 have its natural operator space structure. Let $x \in \mathbb{M}_n(\ell_1)$ with $x = \sum_i x_i \otimes e_i$ (with (e_i) the canonical basis of ℓ_1). Prove

$$\|x\|_{\mathbb{M}_n(\ell_1)} = \sup\big\{\big\|\sum_i x_i \otimes y_i\big\|_{\mathbb{M}_{mn}}\ :\ y_i \in \mathbb{M}_m, \|y_i\| \le 1, m \in \mathbb{N}\big\}.$$

On the other hand, show $S_1[\ell_1] = \ell_1(S_1)$ completely isometrically.

2) Let $1 \le p \le \infty$ and p' be the conjugate index of p. Prove the following completely isometric identities:

$$(C_p)^* \cong C_{p'} \cong R_p \quad \text{and} \quad (R_p)^* \cong R_{p'} \cong C_p$$

by identifying the canonical bases in question.

3) Prove that C_p and R_p are homogeneous Hilbertian spaces.

4) Compute (or estimate) $d_{cb}(C_p^n,\ C_q^n)$ for $1 \le p, q \le \infty$. Then prove that C_p and C_q are not completely isomorphic for $p \ne q$.

6 Noncommutative Khintchine type inequalities

This section is devoted to Khintchine type inequalities in the noncommutative L_p-spaces. These inequalities are of paramount importance in operator space theory and noncommutative analysis.

6.1 The classical Khintchine inequalities. Let (ε_k) be a Rademacher sequence on a probability space (Ω, P), i.e., an independent sequence of random variables such that $P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = 1/2$. The classical Khintchine inequality states that for any $1 \leq p < \infty$ and any finite sequence $(x_n) \subset \mathbb{C}$

$$\big\|\sum_k x_k\varepsilon_k\big\|_p \sim_{c_p} \big\|\sum_k x_k\varepsilon_k\big\|_2 = \big(\sum_k |x_k|^2\big)^{1/2}\,. \tag{6.1}$$

Here as well as in the sequel we use $A \sim_c B$ to abbreviate $c^{-1}B \leq A \leq cB$. c_p denotes a positive constant depending only on p. Using the Fubini theorem, we deduce a similar inequality for coefficients x_n in a commutative L_p-space, say, in $L_p(0,1)$. Namely,

$$\big\|\sum_k x_k\,\varepsilon_k\big\|_{L_p(\Omega;L_p(0,1))} \sim_{c_p} \big\|\big(\sum_k |x_k|^2\big)^{1/2}\big\|_{L_p(0,1)}\,. \tag{6.2}$$

Note that the norm of $L_p(\Omega; L_p(0,1))$ in the above equivalence can be replaced by that of $L_q(\Omega; L_p(0,1))$ for any $1 \leq q < \infty$. (The relevant constant then depends on q too.) This is because of the so-called Khintchine-Kahane inequalities (cf. [Ka]). Let E be a Banach space and $1 \leq p, q < \infty$. Then for any finite sequence $(x_k) \subset E$ we have

$$\big\|\sum_k x_k\,\varepsilon_k\big\|_{L_p(\Omega;E)} \sim_{c_{p,q}} \big\|\sum_k x_k\,\varepsilon_k\big\|_{L_q(\Omega;E)}\,.$$

Now we wish to extend (6.2) to the noncommutative setting, i.e., replacing $L_p(0,1)$ by a noncommutative L_p. We will consider only the case where the coefficients x_k are in the Schatten classes S_p. All inequalities stated below are valid for general noncommutative L_p. On the other hand, we will concentrate mainly on the case $p = 1$ (and the case $p = \infty$ in the free case). Thus our gaol is to find a deterministic expression for

$$\big\|\sum x_k\varepsilon_k\big\|_{L_p(\Omega;S_p)}\,.$$

In view of the square function $\big(\sum |x_k|^2\big)^{1/2}$ in (6.2), we are naturally led to conjecture that the deterministic expression to be found should involve the term

$$\big\|\big(\sum |x_k|^2\big)^{1/2}\big\|_p = \big\|\big(\sum x_k^*x_k\big)^{1/2}\big\|_p\,.$$

This norm is also equal to $\|(x_k)\|_{C_p[S_p]}$. Here and in the sequel, we often identify a sequence (x_k) in E with the element $\sum_k x_k \otimes e_k$ in $C_p[E]$ or $R_p[E]$. But now because of the noncommutativity, we should also take into account the right modulus, i.e., the term

$$\big\|\big(\sum |x_k^*|^2\big)^{1/2}\big\|_p = \big\|\big(\sum x_kx_k^*\big)^{1/2}\big\|_p = \|(x_k)\|_{R_p[S_p]}\,.$$

Although $\|a\|_p = \|a^*\|_p$ for a *single* operator $a \in S_p$, the two terms above are not comparable at all if $p \neq 2$. For example, if $x_k = e_{k1}$, then clearly

$$\big\|\big(\sum_{k=1}^n x_k^*x_k\big)^{1/2}\big\|_p = n^{1/2} \quad\text{and}\quad \big\|\big(\sum_{k=1}^n x_kx_k^*\big)^{1/2}\big\|_p = n^{1/p}\,.$$

6.2 Column and row subspaces. In this subsection we collect some basic properties of the column and row subspaces $C_p[S_p]$ and $R_p[S_p]$.

Proposition 6.1 i) *The Hölder inequality: Let $1 \le p, q, r \le \infty$ such that $1/r = 1/p + 1/q$. Then for any sequences $(x_k) \in C_p[S_p]$ and $(y_k) \in C_q[S_q]$ the series $\sum_k y_k^* x_k$ converges in S_r (with respect to the w^*-topology if $r = \infty$) and*

$$\big\| \sum_k y_k^* x_k \big\|_r \le \big\|(x_k)\big\|_{C_p[S_p]} \big\|(y_k)\big\|_{C_q[S_q]} .$$

ii) *Complementation: $C_p[S_p]$ and $R_p[S_p]$ are 1-complemented subspaces of $S_p(\ell_2 \otimes \ell_2)$ for any $1 \le p \le \infty$. More precisely, let $P: S_p(\ell_2 \otimes \ell_2) \to S_p(\ell_2 \otimes \ell_2)$ be defined by $P(x) = xe$, where $e = 1 \otimes e_{11}$. Then P is a contractive projection from $S_p(\ell_2 \otimes \ell_2)$ onto $C_p[S_p]$.*

iii) *Duality: Let $1 \le p < \infty$, and let p' be the conjugate index of p. Then*

$$C_p[S_p]^* = C_{p'}[S_{p'}] \quad \text{and} \quad R_p[S_p]^* = R_{p'}[S_{p'}]$$

isometrically with respect to the anti-linear duality bracket:

$$\langle (x_k), (y_k) \rangle \mapsto \sum_{k \ge 1} \mathrm{Tr}(y_k^* x_k).$$

Proof. i) Given $(x_k) \subset S_p$ define

$$T((x_k)) = \begin{pmatrix} x_1 & 0 & \cdots \\ x_2 & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix}.$$

We view $T((x_k))$ as an element in $S_p[S_p] = S_p(\ell_2 \otimes \ell_2)$. Now let $(y_k) \subset S_q$ be another finite sequence. Then

$$\sum_k y_k^* x_k = T((y_k))^* \, T((x_k)).$$

So the desired inequality follows from the Hölder inequality in $S_p(\ell_2 \otimes \ell_2)$.

ii) This is obvious. Note that $P(x)$ is the matrix whose first column is that of x and all others are zero.

iii) Using i) one sees that the duality is well-defined. The two duality equalities are immediate consequences of ii) and the duality between $S_p(\ell_2 \otimes \ell_2)$ and $S_{p'}(\ell_2 \otimes \ell_2)$. □

Proposition 6.2 *Let $1 \le p \le \infty$ and $(x_k) \subset S_p$ be a finite sequence.*

i) *If $2 \le p \le \infty$, then*

$$\max\big\{\big\|(x_k)\big\|_{C_p[S_p]},\ \big\|(x_k)\big\|_{R_p[S_p]}\big\} \le \big(\sum_k \|x_k\|_p^2\big)^{1/2} .$$

ii) *If $1 \le p \le 2$, then*

$$\min\big\{\big\|(x_k)\big\|_{C_p[S_p]},\ \big\|(x_k)\big\|_{R_p[S_p]}\big\} \ge \big(\sum_k \|x_k\|_p^2\big)^{1/2} .$$

Proof. i) Let $p \ge 2$. By the triangle inequality in $S_{p/2}$ we have

$$\big\|(x_k)\big\|_{C_p[S_p]}^2 = \big\| \sum_k |x_k|^2 \big\|_{p/2} \le \sum_k \|x_k\|_p^2 .$$

Passing to adjoints we get the inequality on the row norm.

ii) This follows from i) by duality. □

Corollary 6.3 i) *Let* (Σ,μ) *be a measure space, and let* f *belong to the algebraic tensor product* $L_2(\Sigma)\otimes S_p$. *Then if* $2\le p\le\infty$,

$$\max\big\{\big\|\big[\int_\Sigma f(t)^*f(t)d\mu(t)\big]^{1/2}\big\|_p,\ \big\|\big[\int_\Sigma f(t)f(t)^*d\mu(t)\big]^{1/2}\big\|_p\big\}\le\big\|f\big\|_{L_2(\Sigma;S_p)}\,.$$

and if $1\le p\le 2$,

$$\big\|f\big\|_{L_2(\Sigma;S_p)}\le\min\big\{\big\|\big[\int_\Sigma f(t)^*f(t)d\mu(t)\big]^{1/2}\big\|_p,\ \big\|\big[\int_\Sigma f(t)f(t)^*d\mu(t)\big]^{1/2}\big\|_p\big\}.$$

ii) *In particular, if* (φ_k) *is an orthonormal sequence in* $L_2(\Sigma)$ *and* (x_k) *a finite sequence in* S_p, *then for* $2\le p\le\infty$

$$\max\big\{\big\|(x_k)\big\|_{C_p[S_p]},\ \big\|(x_k)\big\|_{R_p[S_p]}\big\}\le\big\|\sum_k x_k\varphi_k\big\|_{L_2(\Sigma;S_p)}\,;$$

and for $1\le p\le 2$

$$\big\|\sum_k x_k\varphi_k\big\|_{L_2(\Sigma;S_p)}\le\inf\big\{\big\|(y_k)\big\|_{C_p[S_p]}\ +\ \big\|(z_k)\big\|_{R_p[S_p]}\big\},$$

where the infimum runs over all decompositions $x_k=y_k+z_k$ *with* y_k *and* z_k *in* S_p.

Proof. i) Note that since $f\in L_2(\Sigma)\otimes S_p$, the two integrals

$$\int_\Sigma f(t)^*f(t)d\mu(t)\quad\text{and}\quad\int_\Sigma f(t)f(t)^*d\mu(t)$$

are well-defined and belong to $S_{p/2}$. The desired inequalities follow from the corresponding ones in the previous proposition.

ii) The first inequality immediately follows from i) above. To prove the second take a decomposition $x_k=y_k+z_k$. Then again by the previous proposition

$$\begin{aligned}\big\|\sum_k x_k\varphi_k\big\|_{L_2(\Sigma;S_p)}&\le\big\|\sum_k y_k\varphi_k\big\|_{L_2(\Sigma;S_p)}+\big\|\sum_k z_k\varphi_k\big\|_{L_2(\Sigma;S_p)}\\&\le\big\|(y_k)\big\|_{C_p[S_p]}+\big\|(z_k)\big\|_{R_p[S_p]};\end{aligned}$$

whence the desired inequality. □

Recall that (C,R) is considered as a compatible pair by identifying their canonical bases with that of ℓ_2. This identification is extended to all spaces C_p and R_p. Thus any pair (C_p,R_q) is also compatible. Then the maximum and infimum in Corollary 6.3, ii) are respectively $\|(x_k)\|_{C_p[S_p]\cap R_p[S_p]}$ and $\|(x_k)\|_{C_p[S_p]+R_p[S_p]}$. For notational simplicity, we introduce the following

Definition 6.4 For $1\le p\le\infty$ define

$$CR_p[S_p]=C_p[S_p]\cap R_p[S_p]\ \text{if}\ p\ge 2\quad\text{and}\quad CR_p[S_p]=C_p[S_p]+R_p[S_p]\ \text{if}\ p<2.$$

Since $C_p[S_p]=S_p[C_p]$ and $R_p[S_p]=S_p[R_p]$,

$$C_p[S_p]\cap R_p[S_p]=S_p[C_p\cap R_p]\quad\text{and}\quad C_p[S_p]+R_p[S_p]=S_p[C_p+R_p]$$

with equivalent norms; moreover, the equivalence constants are controlled by a universal one. By Corollary 5.2, these identities are also completely isomorphic. Thus if we define $CR_p=C_p\cap R_p$ for $p\ge 2$ and $CR_p=C_p+R_p$ for $p<2$, we obtain $CR_p[S_p]=S_p[CR_p]$ with complete equivalent norms.

We are now well prepared for our noncommutative Khintchine inequalities.

6.3 Rademacher sequences. The following is the noncommutative Khintchine inequality for Rademacher variables (ε_k) due to Lust-Piquard and Pisier.

Theorem 6.5 *Let $1 \le p < \infty$ and (x_k) be a finite sequence in S_p. Then*

$$\big\| \sum_k x_k \varepsilon_k \big\|_{L_p(\Omega;S_p)} \sim_{c_p} \big\|(x_k)\big\|_{CR_p[S_p]}\,.$$

More precisely,

i) *if $2 \le p < \infty$, then*

$$\big\|(x_k)\big\|_{CR_p[S_p]} \le \big\| \sum_k x_k \varepsilon_n \big\|_{L_p(\Omega;S_p)} \le c\sqrt{p}\,\big\|(x_k)\big\|_{CR_p[S_p]}\,;$$

ii) *if $1 \le p < 2$, then*

$$c^{-1}\,\big\|(x_k)\big\|_{CR_p[S_p]} \le \big\| \sum_k x_k \varepsilon_k \big\|_{L_p(\Omega;S_p)} \le \big\|(x_k)\big\|_{CR_p[S_p]}\,,$$

where c is a universal positive constant.

Proof. The lower estimate in i) and the upper estimate in ii) follow from Corollary 6.3. We will prove only the lower estimate in ii) for $p = 1$, following the recent approach of Haagerup and Musat [HM]. The reader is referred to [LPP] for the proof of all remaining cases and to [P2] for the optimal order of the best constant for the upper estimate in i).

By duality, the lower estimate of ii) for $p = 1$ is equivalent to the following statement:

($*$) For any finite sequence $(x_k) \subset S_\infty$ there exists a function $f \in L_\infty(\Omega;S_\infty)$ such that

$$\widehat{f}(\varepsilon_k) = x_k \quad \text{and} \quad \|f\|_{L_\infty(\Omega;S_\infty)} \le c\,\big\|(x_k)\big\|_{CR_\infty[S_\infty]},$$

where $\widehat{f}(\varepsilon_k) = \mathbb{E}(f\varepsilon_k)$ with $\mathbb{E}$ denoting the expectation on the probability space (Ω, P).

Let $x_k \in S_\infty$ be selfadjoint and such that $\big\|(x_k)\big\|_{CR_\infty[S_\infty]} \le 1$. Let

$$g = \sum_k \varepsilon_k x_k\,.$$

Then

$$g^2 = \sum_k x_k^2 + \sum_{j<k} \varepsilon_j \varepsilon_k (x_j x_k + x_k x_j)$$

and

$$\begin{aligned} \mathbb{E}(g^4) &= \big(\sum_k x_k^2\big)^2 + \sum_{j<k} (x_j x_k + x_k x_j)^2 \\ &\le 1 + 2\sum_{j<k} (x_j x_k^2 x_j + x_k x_j^2 x_k) \\ &= 1 + 2\sum_j x_j \big(\sum_{k \ne j} x_k^2\big) x_j \le 3. \end{aligned}$$

Now let λ be a positive number to be determined later. Set

$$f_\lambda = g\mathbb{1}_{[-\lambda,\,\lambda]}(g) \quad \text{and} \quad g_\lambda = g - f_\lambda\,,$$

where $\mathbb{1}_{[-\lambda,\,\lambda]}(g)$ denotes the spectral projection of g corresponding to the interval $[-\lambda,\ \lambda]$. By definition, $\|f_\lambda\|_{L_\infty(\Omega;S_\infty)} \le \lambda$. On the other hand, letting $z_k = \mathbb{E}(g_\lambda \varepsilon_k)$, we have

$$\sum_k z_k^2 \le \mathbb{E}(g_\lambda^2).$$

However,

$$\lambda^2 g_\lambda^2 \le g^4 \, .$$

Therefore, combining the preceding inequalities, we find

$$\big\| \big(\sum_k z_k^2 \big)^{1/2} \big\|_\infty \le \big\| \big(\mathbb{E}(g_\lambda^2) \big)^{1/2} \big\|_{L_\infty(\Omega; S_\infty)} \le \lambda^{-1} \big\| \big(\mathbb{E}(g^4) \big)^{1/2} \big\|_{L_\infty(\Omega; S_\infty)} \le \frac{\sqrt{3}}{\lambda} \, .$$

For $\lambda = 2\sqrt{3}$ let

$$f^{(0)} = f_\lambda \, , \quad x_k^{(0)} = \mathbb{E}(f^{(0)} \varepsilon_k), \quad z_k^{(0)} = z_k.$$

Then

$$x_k = x_k^{(0)} + z_k^{(0)} \, , \quad \| f^{(0)} \|_{L_\infty(\Omega; S_\infty)} \le 2\sqrt{3} \, , \quad \big\| \big(z_k^{(0)} \big) \big\|_{CR_\infty[S_\infty]} \le \frac{1}{2} \, .$$

Repeating the same argument with $2z_k^{(0)}$ instead of x_k, we find a function $f^{(1)}$ and a finite sequence $\big(z_k^{(1)} \big)$ such that

$$2z_k^{(0)} = x_k^{(1)} + z_k^{(1)} \, , \quad \| f^{(1)} \|_{L_\infty(\Omega; S_\infty)} \le 2\sqrt{3} \, , \quad \big\| \big(z_k^{(1)} \big) \big\|_{CR_\infty[S_\infty]} \le \frac{1}{2} \, ,$$

where

$$x_k^{(1)} = \mathbb{E}(f^{(1)} \varepsilon_k) \, .$$

Continuing this procedure we obtain a sequence $\big(f^{(n)} \big)$ of functions and a sequence $\big(x_k^{(n)} \big)$ in S_∞ such that

$$x_k = \sum_{n \ge 0} 2^{-n} x_k^{(n)} \, , \quad \| f^{(n)} \|_{L_\infty(\Omega; S_\infty)} \le 2\sqrt{3} \, , \quad x_k^{(n)} = \mathbb{E}(f^{(n)} \varepsilon_k) \, .$$

Put

$$f = \sum_{n \ge 0} 2^{-n} f^{(n)} \, .$$

Then

$$\| f \|_{L_\infty(\Omega; S_\infty)} \le 4\sqrt{3} \quad \text{and} \quad x_k = \mathbb{E}(f \varepsilon_k) \, .$$

Thus the statement $(*)$ is proved for selfadjoint x_k. The general case is easily reduced to the selfadjoint case by decomposing each x_k into its real and imaginary parts. Note that the final constant c obtained in this way is $8\sqrt{3}$. We refer to [HM] for a more careful argument which yields a constant $\sqrt{3}$. ☐

Remark 6.6 Let $\mathcal{R}_p$ be the closed subspace of $L_p(\Omega)$ generated by the ε_k. Using Corollary 5.2, we can rephrase Theorem 6.5 as the fact that $\mathcal{R}_p$ is completely isomorphic to CR_p. This remark also applies to Theorems 6.8 and 6.9 below.

Remark 6.7 The Rademacher sequence (ε_k) can be replaced by a standard Gaussian sequence. More generally, let (φ_k) be an independent sequence of random variables on (Ω, P). Assume that (φ_k) is symmetric in the sense that (φ_k) has the same distribution as $(\pm\varphi_k)$ for any sequence of signs. Assume further that

$$0 < \inf_k \| \varphi_k \|_p, \quad \sup_k \| \varphi_k \|_2 < \infty \quad \text{for } 1 \le p < 2$$

and

$$0 < \inf_k \| \varphi_k \|_2, \quad \sup_k \| \varphi_k \|_p < \infty \quad \text{for } 2 \le p < \infty.$$

Then Theorem 6.5 holds for (φ_k) instead of (ε_k) with relevant constants depending on (φ_k).

Since now we are in the noncommutative setting, it is natural to consider Khintchine type inequalities for noncommutative random variables instead of (ε_k). We first consider the tracial case by giving two important examples. Namely, free generators and semicircular systems. We start with free generators.

6.4 Free generators. Consider a discrete group G. Let $(\delta_g)_{g\in G}$ be the canonical basis of $\ell_2(G)$, i.e., δ_g is the function on G that takes value 1 at g and zero elsewhere. Let $\lambda:\ G\to B(\ell_2(G))$ be the left regular representation. Namely, for any $g\in G$, $\lambda(g)$ is the unitary operator on $\ell_2(G)$ defined by

$$\big(\lambda(g)\varphi\big)(h)=\varphi(g^{-1}h),\quad h,g\in G,\ \varphi\in\ell_2(G).$$

Note that $\lambda(g)\delta_h=\delta_{gh}$ for all $g,h\in G$. Then the group von Neumann algebra $VN(G)$ is the von Neumann subalgebra of $B(\ell_2(G))$ generated by $\{\lambda(g)\ :\ g\in G\}$. Namely, $VN(G)$ is the w*-closure of all finite sums $\sum\alpha_g\lambda(g)$ with $\alpha_g\in\mathbb{C}$. $VN(G)$ also coincides with the left convolution algebra of $\ell_2(G)$. Recall that if $\varphi,\psi\in\ell_2(G)$, their convolution is defined by

$$\varphi*\psi(g)=\sum_{h\in G}\varphi(h)\psi(h^{-1}g),\quad g\in G.$$

Then $x\in VN(G)$ iff there exists $\varphi\in\ell_2(G)$ such that $x\psi=\varphi*\psi$ for every $\psi\in\ell_2(G)$. Let τ_G be the vector state on $VN(G)$ determined by δ_e, i.e., $\tau_G(x)=\langle x\delta_e,\ \delta_e\rangle$ for any $x\in VN(G)$. Then it is easy to check that τ_G is faithful and tracial.

If we identify an operator $x\in VN(G)$ with its symbol $x\delta_e$ in $\ell_2(G)$, then $L_2(VN(G))$ is nothing but $\ell_2(G)$. $L_1(VN(G))$ is traditionally called the Fourier algebra of G and denoted by $A(G)$. Since an operator in $L_1(VN(G))$ is a product of two operators in $L_2(VN(G))$, a function φ on G belongs $A(G)$ iff there exist two functions $\psi,\ \rho\in\ell_2(G)$ such that $\varphi=\psi\,\rho$. We refer to [KR] for more information.

If G is abelian, $VN(G)$ ie equal to $L_\infty(\widehat{G})$, so is commutative, where $\widehat{G}$ is the dual group of G.

An important example of non abelian groups is a free group $\mathbb{F}$ on n generators (g_k) with $n\in\mathbb{N}\cup\{\infty\}$. The sequence $(\lambda(g_k))_k$ of the unitary operators given by the generators is of particular interest. With a slight abuse of terminology, we will also call it a *sequence of free generators.* Note that $(\lambda(g_k))_k$ is orthonormal in $L_2(VN(\mathbb{F}))$. Let us determine its linear span in $VN(\mathbb{F})$. To this end let F_k be the closed subspace of $\ell_2(\mathbb{F})$ generated by all those basic vectors δ_g for which g is a reduced word starting with g_k^{-1}. The F_k, $k=1,2,...$, are mutually orthogonal. Let $\mathcal{F}_k$ be the orthogonal projection from $\ell_2(\mathbb{F})$ onto F_k. Now, given a finite sequence $(\alpha_k)\subset\mathbb{C}$ write

$$\sum_k\alpha_k\lambda(g_k)=\sum_k\alpha_k\lambda(g_k)\mathcal{F}_k+\sum_k\alpha_k\lambda(g_k)\mathcal{F}_k^\perp\,.$$

By the mutual orthogonality of the $\mathcal{F}_k$, we have

$$\begin{aligned}\big\|\sum_k\alpha_k\lambda(g_k)\mathcal{F}_k\big\|_\infty^2&=\big\|(\sum_k\alpha_k\lambda(g_k)\mathcal{F}_k)(\sum_k\alpha_k\lambda(g_k)\mathcal{F}_k)^*\big\|_\infty\\&=\big\|\sum_k|\alpha_k|^2\lambda(g_k)\mathcal{F}_k\lambda(g_k)^*\big\|_\infty\le\sum_k|\alpha_k|^2\,.\end{aligned}$$

To treat another term, observe that for $k\neq j$ the range of $\lambda(g_k)^*\lambda(g_j)\mathcal{F}_j^\perp$ is contained in F_k, so $\mathcal{F}_k^\perp\lambda(g_k)^*\lambda(g_j)\mathcal{F}_j^\perp=0$. Then it follows that

$$\begin{aligned}\big\|\sum_k\alpha_k\lambda(g_k)\mathcal{F}_k^\perp\big\|_\infty^2&=\big\|(\sum_k\alpha_k\lambda(g_k)\mathcal{F}_k^\perp)^*(\sum_k\alpha_k\lambda(g_k)\mathcal{F}_k^\perp)\big\|_\infty\\&=\big\|\sum_k|\alpha_k|^2\mathcal{F}_k^\perp\big\|_\infty\le\sum_k|\alpha_k|^2\,.\end{aligned}$$

Therefore,

$$\big\|\sum_k \alpha_k\lambda(g_k)\big\|_\infty \le 2\big(\sum_k |\alpha_k|^2\big)^{1/2}.$$

The converse inequality with constant 1 is obvious for

$$\big\|\sum_k \alpha_k\lambda(g_k)\big\|_\infty \ge \big\|\sum_k \alpha_k\lambda(g_k)\big\|_2 = \big(\sum_k |\alpha_k|^2\big)^{1/2}.$$

Consequently, for any $1 \le p \le \infty$ the closed subspace generated by the $\lambda(g_k)$ in $L_p(VN(\mathbb{F}))$ is isomorphic to ℓ_2. More precisely, for any finite sequence $(\alpha_k) \subset \mathbb{C}$,

$$\big\|\sum_k \alpha_k\lambda(g_k)\big\|_p \sim_c \big(\sum_k |\alpha_k|^2\big)^{1/2},$$

where c is a universal constant. This inequality remains true if the scalar coefficients (α_k) are replaced by operator coefficients. The resulting inequalities are the Khintchine inequality for free generators. In the following statement $L_p(B(\ell_2)\bar{\otimes}VN(\mathbb{F}))$ is the noncommutative L_p-space associated with the von Neumann tensor product $B(\ell_2)\bar{\otimes}VN(\mathbb{F})$, equipped with the tensor trace $\mathrm{Tr}\otimes\tau_{\mathbb{F}}$, which is a normal semifinite faithful trace. Note that

$$L_p(B(\ell_2)\bar{\otimes}VN(\mathbb{F})) = S_p[L_p(VN(\mathbb{F}))].$$

The following theorem is due to Haagerup/Pisier [HP] for $p = \infty, 1$ and to Pisier [P2] for $1 < p < \infty$.

Theorem 6.8 *Let $1 \le p \le \infty$ and (x_k) be a finite sequence in S_p. Then*

$$\big\|\sum_k x_k\otimes\lambda(g_k)\big\|_{L_p(B(\ell_2)\bar{\otimes}VN(\mathbb{F}))} \sim_c \big\|(x_k)\big\|_{CR_p[S_p]} \tag{6.3}$$

with a universal constant c. Moreover, the closed subspace of $L_p(VN(\mathbb{F}))$ generated by the $\lambda(g_k)$ is completely complemented in $L_p(VN(\mathbb{F}))$ with relevant constant ≤ 2.

Proof. Here we prove (6.3) only in the cases $p = \infty, 1$ and the complementation assertion. The remaining cases are postponed to subsection 6.6. For $p = \infty$ we have the following inequalities

$$\big\|(x_k)\big\|_{CR_\infty[S_\infty]} \le \big\|\sum_k x_k\otimes\lambda(g_k)\big\|_{B(\ell_2)\bar{\otimes}VN(\mathbb{F})} \le 2\big\|(x_k)\big\|_{CR_\infty[S_\infty]} \tag{6.4}$$

for any finite sequence $(x_k) \subset S_\infty$. The proof of the second inequality above is the same as in the scalar case. Let us show the first one. Take a unit vector $\xi \in \ell_2$. Then

$$\begin{aligned}\big\|\sum_k x_k\otimes\lambda(g_k)\big\|_\infty^2 &\ge \langle\sum_k x_k\otimes\lambda(g_k)(\xi\otimes\delta_e),\ \sum_k x_k\otimes\lambda(g_k)(\xi\otimes\delta_e)\rangle\\ &= \langle\sum_k x_k(\xi)\otimes\delta_{g_k},\ \sum_k x_k(\xi)\otimes\delta_{g_k}\rangle\\ &= \sum_k\langle x_k(\xi),\ x_k(\xi)\rangle = \langle\sum_k x_k^*x_k(\xi),\ \xi\rangle;\end{aligned}$$

whence

$$\big\|\big(\sum_k x_k^*x_k\big)^{\frac12}\big\|_\infty \le \big\|\sum_k x_k\otimes\lambda(g_k)\big\|_\infty.$$

Taking adjoints, we find

$$\big\|\big(\sum_k x_kx_k^*\big)^{\frac12}\big\|_\infty \le \big\|\sum_k x_k\otimes\lambda(g_k)\big\|_\infty.$$

Therefore, the lower estimate of (6.4) is proved.

Dualizing (6.4), we get the case $p = 1$:

$$\frac{1}{2}\,\big\|(x_k)\big\|_{CR_1[S_1]} \le \big\|\sum_k x_k \otimes \lambda(g_k)\big\|_{L_1(B(\ell_2)\bar{\otimes}VN(\mathbb{F}))} \le \big\|(x_k)\big\|_{CR_1[S_1]} \tag{6.5}$$

for any finite sequence $(x_k) \subset S_1$. Indeed, let $(y_k) \subset S_\infty$ be such that

$$\big\|(x_k)\big\|_{CR_1[S_1]} = \sum_k \mathrm{Tr}(y_k^* x_k) \quad \text{and} \quad \big\|(y_k)\big\|_{CR_\infty[S_\infty]} \le 1$$

(see Proposition 6.1 iii)). Set

$$x = \sum_k x_k \otimes \lambda(g_k) \in L_1(B(\ell_2)\bar{\otimes}VN(\mathbb{F})) \quad \text{and} \quad y = \sum_k y_k \otimes \lambda(g_k) \in B(\ell_2)\bar{\otimes}VN(\mathbb{F}).$$

Then by (6.4)

$$\sum_k \mathrm{Tr}(y_k^* x_k) = \mathrm{Tr} \otimes \tau_{\mathbb{F}}(y^* x) \le \|y\|_\infty\, \|x\|_1 \le 2\|(y_k)\|_{CR_\infty[S_\infty]}\, \|x\|_1 \le 2\|x\|_1\,.$$

This is the lower estimate of (6.5). To show the upper estimate we consider the projection P from the algebra of all finite sums $\sum \alpha_g \lambda(g)$ with $\alpha_g \in \mathbb{C}$ onto the linear span of the $\lambda(g_k)$. Let $Q = I - P$. It is easy to see that $P(x)$ and $Q(x)$ are orthogonal relative to the scalar product of $L_2(VN(\mathbb{F}))$. Thus the extension of P on $L_2(VN(\mathbb{F}))$ is the orthogonal projection from $L_2(VN(\mathbb{F}))$ onto the closed subspace generated by the $\lambda(g_k)$. Now let $x = \sum_{g\in\mathbb{F}} x_g \otimes \lambda(g)$ be a finite sum with $x_g \in S_\infty$. Write

$$x = \mathrm{id}_{S_\infty} \otimes P(x) + \mathrm{id}_{S_\infty} \otimes Q(x) = \sum_k x_k \otimes \lambda(g_k) + \mathrm{id}_{S_\infty} \otimes Q(x).$$

Then using the argument yielding the first inequality of (6.4), we get

$$\big\|(x_k)\big\|_{CR_\infty[S_\infty]} \le \|x\|_\infty\,. \tag{6.6}$$

Together with a duality argument as above, this inequality implies the upper estimate of (6.5).

Combining (6.4) and (6.6), we get

$$\|P(x)\|_\infty \le 2\|x\|_\infty\,.$$

This implies that P extends to a completely bounded normal projection on $VN(\mathbb{F})$. Indeed, by the density of all finite sums $\sum_{g\in\mathbb{F}} x_g \otimes \lambda(g)$, $x_g \in S_1$, in $L_1(B(\ell_2)\bar{\otimes}VN(\mathbb{F}))$, we find that (6.5) shows that P admits an extension on $L_1(VN(\mathbb{F}))$ which is completely bounded with cb-norm ≤ 2. Taking adjoint, we obtain the desired completely bounded normal extension of P on $VN(\mathbb{F})$. The complete boundedness of P on $L_p(VN(\mathbb{F}))$ is then proved by interpolation. □

6.5 Semicircular systems. Let H be a complex Hilbert space. The associated free (or full) Fock space is defined by

$$\mathcal{F}(H) = \bigoplus_{n\ge 0} H^{\otimes n},$$

where $H^{\otimes 0} = \mathbb{C}\mathbb{1}$ ($\mathbb{1}$ being a unit vector, called vacuum), and $H^{\otimes n}$ is the n-th Hilbertian tensor power of H for $n \ge 1$. The (left) creator associated with a vector $\xi \in H$ is the operator on $\mathcal{F}(H)$ uniquely determined by

$$c(\xi)\,\eta_1 \otimes \cdots \otimes \eta_n = \xi \otimes \eta_1 \otimes \cdots \otimes \eta_n$$

for any $\eta_1, ..., \eta_n \in H$. Here $\eta_1 \otimes \cdots \otimes \eta_n$ is understood as the vacuum $\mathbb{1}$ if $n = 0$. It is clear that $c(\xi)$ is bounded and $\|c(\xi)\| = \|\xi\|$. The adjoint of $c(\xi)$ is given by

$$c(\xi)^* \, \eta_1 \otimes \cdots \otimes \eta_n = \langle \eta_1, \ \xi \rangle \, \eta_2 \otimes \cdots \otimes \eta_n$$

for any $\eta_1, ..., \eta_n \in H$ with $n \geq 1$ and $c(\xi)^* \mathbb{1} = 0$. This is the annihilator associated with ξ and is denoted by $a(\xi)$. Note that the map $\xi \mapsto c(\xi)$ is linear, while $\xi \mapsto a(\xi)$ is anti-linear. We have the following free commutation relation:

$$a(\eta)c(\xi) = \langle \xi, \ \eta \rangle 1, \quad \forall \, \xi, \ \eta \in H. \tag{6.7}$$

Now assume that H is the complexification of a real Hilbert space $H_{\mathbb{R}}$. The vectors $H_{\mathbb{R}}$ are called *real.* Let $\xi \in H$ be real and define

$$s(\xi) = c(\xi) + a(\xi).$$

$s(\xi)$ is a *semicircular element* in Voiculescu's sense. We will also call it a *free Gaussian variable.* Note that the map $\xi \mapsto s(\xi)$ is real linear from $H_{\mathbb{R}}$ into $B(\mathcal{F}(H))$. Then the *free von Neumann algebra* $\Gamma(H)$ associated with H is the von Neumann subalgebra of $B(\mathcal{F}(H))$ generated by all $s(\xi)$ with real $\xi \in H$:

$$\Gamma_0(H) = \big\{ s(\xi) : \xi \in H_{\mathbb{R}} \big\}'' \subset B(\mathcal{F}(H))\,.$$

The vector state τ_0 defined by the vacuum, $x \mapsto \langle x\mathbb{1}, \ \mathbb{1} \rangle$ is faithful and tracial on $\Gamma_0(H)$. We refer to [VDN] for more information.

Let us mention some basic properties of free Gaussian variables. Let $\xi \in H$ be a unit real vector. By definition, $s(\xi)$ is selfadjoint; its spectrum is $[-2, 2]$ and the corresponding measure induced by τ_0 is the so-called Wigner measure

$$d\mu(t) = \frac{1}{2\pi} \sqrt{4 - t^2} \, dt.$$

Thus for any $1 \leq p < \infty$

$$\big\| s(\xi) \big\|_p = \Big[\int_{-2}^{2} |t|^p \sqrt{4 - t^2} \, \frac{dt}{2\pi} \Big]^{1/p} .$$

Therefore,

$$\big\| s(\xi) \big\|_\infty = 2, \quad \big\| s(\xi) \big\|_2 = 1, \quad \big\| s(\xi) \big\|_1 = \frac{8}{3\pi} \,.$$

Now let (ξ_k) be an orthonormal sequence of real vectors of H. $(s(\xi_k))$ is then called a *standard semicircular system.* Like the classical standard Gaussian variables, $(s(\xi_k))$ has the following invariance property. Let $(\alpha_k) \subset \mathbb{R}$ be such that $\sum_k |\alpha_k|^2 = 1$. Then $\sum_n \alpha_k s(\xi_k)$ (convergence in the strong operator topology) is again a free Gaussian variable, i.e., $s(\xi)$ with $\xi = \sum_k \alpha_k \, \xi_k$. More generally, if $s(\xi_1), ..., s(\xi_n)$ is a standard semi-circular system and if $u = (u_{jk})$ is an orthogonal $n \times n$ matrix, then

$$\big(\sum_{k=1}^{n} u_{jk} s(\xi_k) \big)_{1 \leq j \leq n}$$

is still a standard semi-circular system. Indeed,

$$\sum_{k=1}^{n} u_{jk} s(\xi_k) = s(\sum_{k=1}^{n} u_{jk} \, \xi_k) = s(\eta_j),$$

where $\eta_j = \sum_k u_{jk} \, \xi_k$. The η_j form an orthonormal family. This orthogonal invariance implies that if $(s(\xi_k))$ is a standard semicircular system, then for $(\alpha_k) \subset \mathbb{R}$ such that $\sum_k |\alpha_k|^2 < \infty$ and any $1 \leq p \leq \infty$

$$\big\| \sum_{k \geq 1} \alpha_k \, s(\xi_k) \big\|_p = \big(\sum_{k \geq 1} |\alpha_k|^2 \big)^{1/2} \, \big\| s(\xi) \big\|_p \,.$$

In particular, $(s(\xi_k))$ is an orthonormal sequence in $L_2(\Gamma(H))$.

For simplicity we now assume that H is infinite dimensional and separable and put $\Gamma_0 = \Gamma_0(H)$. Let (e_k) be an orthonormal basis of H and set $s_k = s(e_k)$. Thus (s_k) is a semicircular sequence and Γ_0 is generated by the s_k. The following inequality for semicircular systems comes from [P2].

Theorem 6.9 *Let $1 \le p \le \infty$ and (x_k) be a finite sequence in S_p. Then*

$$\big\|\sum_k x_k\otimes s_k\big\|_{L_p(B(\ell_2)\bar\otimes\Gamma_0)} \sim_c \big\|(x_k)\big\|_{CR_p[S_p]}\,. \tag{6.8}$$

Moreover, the closed subspace of $L_p(\Gamma_0)$ generated by the s_k is completely complemented in $L_p(\Gamma_0)$ with constant 2.

Proof. This proof is similar to that of Theorem 6.8. Again, the case $1 < p < \infty$ will be proved later. By the free commutation relation (6.7), we have

$$\big\|\sum_k x_k\otimes c(e_k)\big\|_\infty^2 = \big\|\big(\sum_k x_k\otimes c(e_k)\big)^*\big(\sum_k x_k\otimes c(e_k)\big)\big\|_\infty = \big\|\sum_k x_k^*x_k\big\|_\infty\,.$$

Similarly,

$$\big\|\sum_k x_k\otimes a(e_k)\big\|_\infty^2 = \big\|\sum_k x_kx_k^*\big\|_\infty\,.$$

It follows that

$$\big\|\sum_k x_k\otimes s_k\big\|_\infty \le \big\|\big(\sum_k x_k^*x_k\big)^{\frac12}\big\|_\infty + \big\|\big(\sum_k x_kx_k^*\big)^{\frac12}\big\|_\infty\,.$$

For the lower estimate, take a unit vector $\xi\in\ell_2$. Then

$$\begin{aligned}\big\|\sum_k x_k\otimes s_k\big\|_\infty^2 &\ge \big\langle \sum_k x_k\otimes s_k(\xi\otimes \mathbb{1}),\ \sum_k x_k\otimes s_k(\xi\otimes\mathbb{1})\big\rangle\\ &= \big\langle\sum_k x_k(\xi)\otimes e_k,\ \sum_k x_k(\xi)\otimes e_k\big\rangle\\ &= \big\langle\sum_k x_k^*x_k(\xi),\ \xi\big\rangle;\end{aligned}$$

whence

$$\big\|\big(\sum_k x_k^*x_k\big)^{\frac12}\big\|_\infty \le \big\|\sum_k x_k\otimes s_k\big\|_\infty\,.$$

Similarly,

$$\big\|\big(\sum_k x_kx_k^*\big)^{\frac12}\big\|_\infty \le \big\|\sum_k x_k\otimes s_k\big\|_\infty\,.$$

Thus the lower estimate of (6.8) for $p=\infty$ follows.

(6.8) for $p=1$ and the complementation assertion are proved in the same way as the corresponding assertions of Theorem 6.8. We omit the details. □

Remark 6.10 The free Gaussian variables in the theorem above can be replaced by Bożejko-Speicher's q-Gaussians for $-1 < q < 1$ (see [BKS] and [BS]). This inequality remains also true for the case $q=-1$, for which the corresponding Gaussians become the so-called Fermions. Note that the case $q=1$ corresponds to the classical Gaussian case. In strong contrast with the case $-1<q<1$, the analogue for $q=\pm1$ of Theorem 6.9 fails for $p=\infty$.

6.6 Complete unconditionality. Noncommutative Khintchine inequalities are closely related to complete unconditionality. Let M be a von Neumann algebra equipped with a normal faithful tracial state τ. Let (a_k) be a sequence in $L_p(M)$, $1 \le p \le \infty$. Assume that the noncommutative Khintchine inequality holds for (a_k): for any finite sequence (x_k) in S_p

$$\big\| \sum_k x_k \otimes a_k \big\|_{L_p(B(\ell_2)\bar{\otimes} M)} \sim \big\|(x_k)\big\|_{CR_p[S_p]} .$$

It then follows that the subspace E generated by the a_k in $L_p(M)$ is completely isomorphic to CR_p (see Remark 6.6). Since the canonical basis of CR_p is *completely unconditional*, so is the sequence (a_k). Namely, there exists a constant λ such that

$$\big\| \sum_k \varepsilon_k x_k \otimes a_k \big\|_p \le \lambda \big\| \sum_k x_k \otimes a_k \big\|_p \tag{6.9}$$

for any finite sequence $(x_k) \subset S_p$ and any $\varepsilon_k = \pm 1$. This property can be rephrased as follows. Given any sequence (ε_k) of signs the map $\sum_k \alpha_k a_k \mapsto \sum_k \varepsilon_k \alpha_k a_k$ on E is c.b. with cb-norm $\le \lambda$. Therefore, if the noncommutative Khintchine inequality holds for (a_k), then (a_k) is a completely unconditional basic sequence. The converse is also true with some additional mild conditions for $p < \infty$. We consider here only the case $1 \le p < 2$. The following result is proved independently by the author and Junge/Oikhberg [JO]. The latter paper contains more results of the same type.

Theorem 6.11 *Let M be a von Neumann algebra equipped with a normal faithful tracial state τ. Let $1 \le p < 2$ and $(a_k)_{k\ge 1} \subset L_p(M)$ be a completely unconditional basic sequence with constant λ. Assume that*

$$\delta = \inf_k \|a_k\|_p > 0 \quad \textit{and} \quad \Delta = \sup_k \|a_k\|_2 < \infty .$$

Then the noncommutative Khintchine inequality holds for (a_k) with relevant constants depending only on λ, δ and Δ. More precisely, for any finite sequence $(x_k) \subset S_p$

$$c\,\delta\lambda^{-1} \big\|(x_k)\big\|_{CR_p[S_p]} \le \big\| \sum_k x_k \otimes a_k \big\|_p \le \Delta\lambda \, \big\|(x_k)\big\|_{CR_p[S_p]} , \tag{6.10}$$

where c is an absolute positive constant.

Proof. Let $(x_k) \subset S_p$ be a finite sequence. We then have (6.9). Averaging the left hand side of (6.9) over the ε_n and using the noncommutative Khintchine inequality in Theorem 6.5, we get

$$\inf \Big\{ \big\| \big(\sum_k Y_k^* Y_k\big)^{1/2} \big\|_p + \big\| \big(\sum_k Z_k Z_k^*\big)^{1/2} \big\|_p \Big\} \le c\,\lambda \, \big\| \sum_k x_k \otimes a_k \big\|_p ,$$

where the infimum runs over all decompositions $x_k \otimes a_k = Y_k + Z_k$ with Y_k and Z_k in $L_p(B(\ell_2)\bar{\otimes} M)$. Fix such a decomposition $x_k \otimes a_k = Y_k + Z_k$. Now choose $b_k \in L_{p'}(M)$ such that

$$\tau(a_k b_k) = 1 \quad \text{and} \quad \|b_k\|_{p'} = \|a_k\|_p^{-1} .$$

Then

$$x_k = \mathrm{id} \otimes \tau((x_k \otimes a_k)(1 \otimes b_k)) = \mathrm{id} \otimes \tau(Y_k(1 \otimes b_k)) + \mathrm{id} \otimes \tau(Z_k(1 \otimes b_k)) \overset{\mathrm{def}}{=} y_k + z_k .$$

Note that $\mathrm{id} \otimes \tau$ is the natural conditional expectation from $B(\ell_2)\bar{\otimes} M$ onto $B(\ell_2)$ ($B(\ell_2)$ being viewed as a von Neumann subalgebra of $B(\ell_2)\bar{\otimes} M$ via $x \leftrightarrow x \otimes 1$). Also note that y_k and z_k belong to S_p. We need to majorize $\|(\sum y_k^* y_k)^{1/2}\|_p$ (resp. $\|(\sum z_k z_k^*)^{1/2}\|_p$) by $\|(\sum Y_k^* Y_k)^{1/2}\|_p$

(resp. $\|(\sum Z_kZ_k^*)^{1/2}\|_p$). To this end, let $(u_k)\subset S_{p'}$ be such that $\|(\sum_k u_ku_k^*)^{1/2}\|_{p'}\le 1$. By the Hölder inequality in Proposition 6.1

$$\begin{aligned}\big|\sum_k \mathrm{Tr}(y_ku_k)\big| &= \big|\sum_k \mathrm{Tr}[\mathrm{id}\otimes\tau(Y_k(1\otimes b_k))u_k]\big| = \big|\sum_k \mathrm{Tr}\otimes\tau[Y_k(u_k\otimes b_k)]\big|\\ &\le \big\|\big(\sum_k u_ku_k^*\otimes b_kb_k^*\big)^{1/2}\big\|_{p'}\,\big\|\big(\sum_k Y_k^*Y_k\big)^{1/2}\big\|_p\,.\end{aligned}$$

We claim that

$$\big\|\big(\sum_k u_ku_k^*\otimes b_kb_k^*\big)^{1/2}\big\|_{p'}\le \sup_k\|b_k\|_{p'}\,\big\|\big(\sum_k u_ku_k^*\big)^{1/2}\big\|_{p'}\,.$$

Indeed, this is obvious for $p'=2$ and $p'=\infty$. Then complex interpolation yields the case $2<p'<\infty$. Combining the preceding inequalities, we find

$$\begin{aligned}\big|\sum_k \mathrm{Tr}(y_ku_k)\big| &\le \sup_k\|b_k\|_{p'}\big\|\big(\sum_k u_ku_k^*\big)^{1/2}\big\|_{p'}\,\big\|\big(\sum_k Y_k^*Y_k\big)^{1/2}\big\|_p\\ &\le \delta^{-1}\big\|\big(\sum_k Y_k^*Y_k\big)^{1/2}\big\|_p\,.\end{aligned}$$

Thus taking the supremum over all (u_k), we get

$$\big\|\big(\sum_k y_k^*y_k\big)^{1/2}\big\|_p\le \delta^{-1}\big\|\big(\sum_k Y_k^*Y_k\big)^{1/2}\big\|_p\,.$$

Similarly,

$$\big\|\big(\sum_k z_kz_k^*\big)^{1/2}\big\|_p\le \delta^{-1}\big\|\big(\sum_k Z_kZ_k^*\big)^{1/2}\big\|_p\,.$$

Therefore, we deduce

$$\big\|(x_k)\big\|_{CR_p[S_p]}\le c\,\delta^{-1}\lambda\,\big\|\sum_k x_k\otimes a_k\big\|_p\,.$$

To prove the upper estimate we use again the complete unconditionality of (a_k) and the noncommutative Khintchine inequality. Then we have

$$\big\|\sum_k x_k\otimes a_k\big\|_p\le\lambda\inf_{x_k=y_k+z_k}\big\{\big\|\big(\sum_k y_k^*y_k\otimes a_k^*a_k\big)^{1/2}\big\|_p+\big\|\big(\sum_k z_kz_k^*\otimes a_ka_k^*\big)^{1/2}\big\|_p\big\}.$$

Now our task is to remove $a_k^*a_k$ and $a_ka_k^*$ from the terms on the right. To this end we use the natural conditional expectation $\mathbb{E}$ from $B(\ell_2)\bar\otimes M$ onto $B(\ell_2)$, already mentioned earlier. $\mathbb{E}$ is determined by $\mathbb{E}(x\otimes u)=\tau(u)x\otimes 1\sim\tau(u)x$ for $x\in B(\ell_2)$ and $u\in M$, i.e., $\mathbb{E}=\mathrm{id}\otimes\tau$. $\mathbb{E}$ is normal and faithful. As usual, $\mathbb{E}$ extends to a contractive projection on $L_q(B(\ell_2)\bar\otimes M)$ for every $1\le q<\infty$. For our purpose here we need to consider the case $q\le 1$. We claim that if X is a positive operator in $L_q(B(\ell_2)\bar\otimes M)\cap L_1(B(\ell_2)\bar\otimes M)$ with $q\le 1$, then

$$\|X\|_q\le\|\mathbb{E}(X)\|_q\,. \tag{6.11}$$

This is a consequence of the operator concavity of the map $X\mapsto X^q$ for $0<q\le 1$. Indeed, using Stinespring's dilation theorem and Hansen's inequality [Ha], we deduce that $(\mathbb{E}(X))^q\ge\mathbb{E}(X^q)$ for any positive $X\in B(\ell_2)\bar\otimes M$. This clearly implies (6.11).

Return back to our task. Using (6.11) with $q = p/2$ (recalling that $1 \le p < 2$), we deduce that

$$\begin{aligned}\big\|\big(\sum_k y_k^* y_k \otimes a_k^* a_k\big)^{1/2}\big\|_p^2 &= \big\|\sum_k y_k^* y_k \otimes a_k^* a_k\big\|_{p/2}\\ &\le \big\|\mathbb{E}\big[\sum_k y_k^* y_k \otimes a_k^* a_k\big]\big\|_{p/2}\\ &= \big\|\sum_k y_k^* y_k \otimes \tau(a_k^* a_k)1\big\|_{p/2}\\ &\le \sup_k \|a_k\|_2^2\, \big\|\big(\sum_k y_k^* y_k\big)^{1/2}\big\|_p^2 .\end{aligned}$$

Therefore,

$$\big\|\big(\sum_k y_k^* y_k \otimes a_k^* a_k\big)^{1/2}\big\|_p \le \Delta\, \big\|\big(\sum_k y_k^* y_k\big)^{1/2}\big\|_p .$$

A similar inequality holds for another term involving the z_k. It thus follows that

$$\big\|\sum_k x_k \otimes a_k\big\|_p \le \Delta\,\lambda\, \big\|(x_k)\big\|_{CR_p[S_p]}.$$

This is the desired upper estimate, and so the proof of the theorem is complete. □

End of the proofs of Theorems 6.8 and 6.9. The $\lambda(g_k)$ are unitary, so $\|\lambda(g_k)\|_p = 1$ for any $1 \le p \le \infty$. On the other hand, for any sequence (ε_k) of sings, there exists a unique representation π of $VN(\mathbb{F})$ determined by $\pi(\lambda(g_k)) = \varepsilon_k \lambda(g_k)$. Moreover, π is trace preserving: $\tau_{\mathbb{F}} \circ \pi = \tau_{\mathbb{F}}$. It follows that π extends to a complete isometry on $L_p(VN(\mathbb{F}))$ for every $1 \le p \le \infty$. Therefore, for any finite sequence $(x_k) \subset S_p$ we have

$$\big\|\sum_k \varepsilon_k x_k \otimes \lambda(g_k)\big\|_p = \big\|\sum_k x_k \otimes \lambda(g_k)\big\|_p .$$

Thus the sequence $(\lambda(g_k))$ is completely unconditional with constant 1. Then Theorem 6.11 implies (6.3) for $1 \le p < 2$. The case $p = 2$ is trivial. The case $2 < p < \infty$ is obtained by duality and using the complementation property in Theorem 6.11.

The proof of Theorem 6.9 is similar. This time to get the representation π of Γ_0 such that $\pi(s_k) = \varepsilon_k s_k$, we have to use second quantization (see [VDN]). □

Remark 6.12 There exist many examples satisfying the assumption of Theorem 6.11. This is the case of a sequence of q-Gaussians mentioned in Remark 6.10. In particular, for $q = -1$, we get the noncommutative Khintchine inequality for a sequence of Fermions.

6.7 Generalized circular systems. Remind that we wish to embed OH into a noncommutative L_1-space by using a certain noncommutative Khintchine type inequality. Namely, we have to prove that OH is completely isomorphic to the closed subspace generated by a sequence of random variables in an $L_1(M)$ for a von Neumann algebra M. Such a sequence cannot satisfy the assumption of Theorem 6.11 for OH is not completely isomorphic to CR_1. In fact, Pisier [P5] showed that OH cannot completely embed into an $L_1(M)$ with M semifinite (i.e., of type I or II). This explains why we are forced to seek for Khintchine type inequalities for random variables in a non tracial probability space, i.e., the underlying von Neumann algebra is of type III. We give below only one example of this kind.

Fix an infinite dimensional separable Hilbert space H as in subsection 6.5. Let $\{e_{\pm k}\}_{k\ge 1}$ be an orthonormal basis of H. We also fix a sequence $\{\lambda_k\}_{k\ge1}$ of positive numbers. Let

$$g_k = c(e_k) + \sqrt{\lambda_k}\, a(e_{-k}).$$

$(g_k)_{k\ge 1}$ is a *generalized circular system* in Shlyakhtenko's sense [S]. Let Γ be the von Neumann algebra on the full Fock space $\mathcal{F}(H)$ generated by the g_k. Let ρ be the vector state on Γ determined by the vacuum $\mathbb{1}$. Then ρ is faithful on Γ. By the identification of $L_1(\Gamma)$ with the predual Γ_*, ρ is a positive unit element of $L_1(\Gamma)$, so for any $1\le p\le\infty$, $\rho^{1/p}$ is a positive unit element of $L_p(\Gamma)$, and thus $g_k\,\rho^{1/p}\in L_p(\Gamma)$ for any k.

Shlyakhtenko proved that the algebra Γ is a type III_λ factor $(0<\lambda\le 1)$ if not all λ_k are equal to 1. Γ is not hyperfinite. Recall that Γ is the free analogue of the classical Araki-Woods quasi-free CAR factors. The latter factors are hyperfinite type III_λ.

The following is the noncommutative Khintchine type inequalities for generalized circular systems, proved in [PS] for $p=\infty$ and in [X2] for $p<\infty$.

Theorem 6.13 *Let (x_n) be a finite sequence in S_p, $1\le p\le\infty$.*

i) *If $p\ge 2$,*

$$\Big\|\sum_k x_k\otimes g_k\rho^{\frac1p}\Big\|_{L_p(B(\ell_2)\bar\otimes\Gamma)}\sim_c \max\Big\{\Big\|\Big(\sum_k x_k^*x_k\Big)^{\frac12}\Big\|_p\,,\ \Big\|\Big(\sum_k \lambda_k^{1-\frac2p}\,x_kx_k^*\Big)^{\frac12}\Big\|_p\Big\}.$$

ii) *If $p<2$,*

$$\Big\|\sum_k x_k\otimes g_k\rho^{\frac1p}\Big\|_{L_p(B(\ell_2)\bar\otimes\Gamma)}\sim_c \inf\Big\{\Big\|\Big(\sum_k y_k^*y_k\Big)^{\frac12}\Big\|_p+\Big\|\Big(\sum_k \lambda_k^{1-\frac2p}\,z_kz_k^*\Big)^{\frac12}\Big\|_p\Big\},$$

where the infimum runs over all decompositions $x_k=y_k+z_k$ in S_p.

iii) *Let $\mathcal{G}_p$ be the closed subspace of $L_p(\Gamma)$ generated by $\{g_k\rho^{\frac1p}\}_{k\ge1}$. Then $\mathcal{G}_p$ is completely complemented in $L_p(\Gamma)$ with constant 2.*

Proof. We prove only the cases $p=\infty,1$ and the complementation assertion. The proof of i) for $p=\infty$ is the same as that of (6.4). Let us prove iii), which will imply ii) for $p=1$ by duality. As in the semicircular case, consider the projection P from the algebra of polynomials on the g_k onto the linear span of the g_k. Let $Q=I-P$. It is easy to see that $P(x)$ and $Q(x)$ are orthogonal relative to both scalar products $(a,b)\mapsto\rho(b^*a)$ and $(a,b)\mapsto\rho(ab^*)$. Now let x be a polynomial on the g_k with coefficients in S_∞. Write x as

$$x=\mathrm{id}_{S_\infty}\otimes P(x)+\mathrm{id}_{S_\infty}\otimes Q(x)=\sum_k a_k\otimes g_k+\mathrm{id}_{S_\infty}\otimes Q(x).$$

Then using the argument yielding the first inequality of (6.4), we get

$$\max\Big\{\Big\|\Big(\sum_k a_k^*a_k\Big)^{\frac12}\Big\|_\infty\,,\ \Big\|\Big(\sum_k \lambda_k\,a_ka_k^*\Big)^{\frac12}\Big\|_\infty\Big\}\le\|x\|_\infty\,.$$

Therefore, using i) in the case $p=\infty$, we deduce that P is completely bounded. This implies, in turn, ii) for $p=1$ by duality. Let us also note that P extends to a completely bounded normal projection on Γ. Indeed, by the density of $\{x\rho\ :\ x$ polynomial on the $g_k\}$ in $L_1(\Gamma)$, we find that ii) in the case $p=1$ shows that P admits a pre-adjoint on $L_1(\Gamma)$ which is completely bounded. Finally, by interpolation we obtain iii) for $1<p<\infty$. □

Remark 6.14 In the spirit of Remark 6.6, Theorem 6.11 can be reformulated as a complete isomorphism between the subspace $\mathcal{G}_p$ and a weighted version of CR_p. Given a sequence (μ_k) of positive numbers we denote by $C((\mu_k))$ the weighted ℓ_2-space $\ell_2((\mu_k))$ equipped with the column Hilbert space structure. More generally, for any $p\ge1$ we define $C_p((\mu_k))$ to be the weighted version of C_p. Note that $C_p((\mu_k))$ is completely isometric to C_p and its operator space structure is determined as follows: for any finite sequence $(x_k) \subset S_p$

$$\big\| \sum_k x_k \otimes e_k \big\|_{S_p[C_p((\mu_k))]} = \big\| \big(\sum_k \mu_k \, x_k^* x_k \big)^{\frac{1}{2}} \big\|_{S_p} .$$

Similarly, we define the weighted p-row space $R_p((\mu_k))$. Then Theorem 6.11 implies that $\mathcal{G}_p$ is completely isomorphic to $C_p \cap R_p((\mu_k))$ for $p \geq 2$ and to $C_p + R_p((\mu_k))$ for $p < 2$, where $\mu_k = \lambda_k^{1-2/p}$.

Remark 6.15 Theorem 6.11 admits a Fermionic analogue. In this case the corresponding algebra is an Araki-Woods hyperfinite type III factor. The resulting inequality holds only for $p < \infty$ and the relevant constants depend only on p and blow up as $p \to \infty$. We refer to [J3] and [X1] for more information (see also [HM] for an alternate approach for the case $p = 1$).

Exercices:

1) Prove the classical Khintchine inequality (6.1). (Consider first the case of an even integer p.)

2) Prove that there exists no constant c_p depending on p such that

$$\big\| \sum_k x_k \varepsilon_n \big\|_{L_p(\Omega; S_p)} \leq c_p \, \big\| (x_k) \big\|_{C_p[S_p]} \quad \text{if } p > 2$$

   or

$$\big\| (x_k) \big\|_{C_p[S_p]} \leq c_p \, \big\| \sum_k x_k \varepsilon_n \big\|_{L_p(\Omega; S_p)} \quad \text{if } p < 2$$

   holds for all finite sequences $(x_k) \subset S_p$.

3) Prove Remark 6.7.

4) Let G be a discrete group. Prove that the vector state τ_G determined by δ_e is faithful and tracial on $VN(G)$.

5) Prove that the vector state τ_0 determined by the vacuum $\mathbb{1}$ is faithful and tracial on the free von Neumann algebra $\Gamma_0(H)$.

7 Embedding of OH into noncommutative L_1

Now we use the Khintchine inequality for generalized circular systems to embed OH completely isomorphically into a noncommutative L_1-space, i.e., to show that OH is completely isomorphic to a subspace of a noncommutative L_1. This is a remarkable theorem of Junge [J2]. The following representation of OH as a quotient of a subspace of $C \oplus R$ will be crucial.

Theorem 7.1 *OH is completely isometric to a quotient of a subspace of $C \oplus_\infty R$.*

The direct sum in the ℓ_∞-sense can be replaced by a direct sum in the ℓ_1-sense at the price of a constant 2. Since the complete embeddings in the sequel are only completely isomorphic, we will forget the index ∞ or 1 in all direct sums. The preceding theorem is Exercice 7.8 of [P3]. The proof there gives the following more precise statement.

Theorem 7.2 *There exists an injective positive selfadjoint (unbounded) operator $\Delta : C \to R$ such that OH is completely isometric to a quotient of $G(\Delta)$, where $G(\Delta)$ is the graph of Δ*

$$G(\Delta) = \big\{ (h, \Delta h) \ : \ h \in D(\Delta) \big\} \subset C \oplus R,$$

considered as a subspace of $C \oplus R$.

Recall that both C and R are isometric to ℓ_2 as Banach spaces. Thus Δ is an operator on ℓ_2 with domain $D(\Delta)$. Since Δ is positive and injective, its range is also dense. Passing to duality, we see that OH^* is completely isometric to a subspace of $G(\Delta)^*$. Since $\overline{OH^*} = OH$ completely isometrically, embedding OH into an L_1 is reduced to embedding $G(\Delta)^*$ into an L_1.

To see why Khintchine type inequalities can help us for such a matter, let us describe the operator space structure of $G(\Delta)$:

$$G(\Delta) = \{(h,\ \Delta(h))\ :\ h \in D(\Delta)\} \subset C \oplus_\infty R\,.$$

Let $x_k \in S_\infty$ and $h_k \in D(\Delta)$. Let $x = \sum_k x_k \otimes (h_k,\ \Delta(h_k)) \in S_\infty \otimes_{\min} G(\Delta)$. Then

$$\|x\|_{S_\infty \otimes_{\min} G(\Delta)} = \max\big\{\big\|\sum_{k,j}\langle h_k,\ h_j\rangle x_j^* x_k\big\|_{S_\infty}^{1/2},\ \big\|\sum_{k,j}\langle \Delta(h_k),\ \Delta(h_j)\rangle x_k x_j^*\big\|_{S_\infty}^{1/2}\big\}.$$

This is a continuous version of the space (for $p = \infty$) introduced in Remark 6.14. To simplify our discussion and without loss of generality by a simple argument of approximation, we may assume that Δ has pure point spectrum, i.e., ℓ_2 has an orthonormal basis (e_k) of eigenvectors of Δ. Let (μ_k) be the associated eigenvalues: $\Delta e_k = \mu_k e_k$. It then follows that $G(\Delta)$ coincides (completely isometrically) with the diagonal subspace $C \cap R((\lambda_k))$ of $C \oplus_\infty R((\lambda_k))$, where $\lambda_k = \mu_k^2$.

By duality, we deduce that $G(\Delta)^* = C_1 + R_1((\lambda_k^{-1}))$. More precisely, the operator space structure of $G(\Delta)^*$ is determined as follows: for any finite sequence $(x_k) \subset S_1$:

$$\|x\|_{S_1[G(\Delta)^*]} = \inf\big\{\big\|\big(\sum_k y_k^* y_k\big)^{1/2}\big\|_{S_1} + \big\|\big(\sum_k \lambda_k^{-1} z_k z_k^*\big)^{1/2}\big\|_{S_1}\big\},$$

where the infimum runs over all decompositions $x_k = y_k + z_k$ in S_1. Therefore, by the Khintchine inequality in Theorem 6.13 with $p = 1$, $G(\Delta)^*$ is completely isomorphic to $\mathcal{G}_1$ there. Thus we have proved the following

Theorem 7.3 *OH is completely isomorphic to a subspace of a noncommutative L_1-space.*

Remark 7.4 i) The proof of Theorem 7.3 gives much more. In fact, it shows that the dual space of any graph in $C \oplus R$ completely embeds into an L_1. From this one can deduce that a quotient of a subspace of $C \oplus R$ completely embeds into an L_1. We refer to [J2], [P4] and [X1] for more information.

ii) The von Neumann algebra Γ is not hyperfinite. OH also completely embeds into the predual of a hyperfinite algebra (see [J3], [HM]).

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