

**Lectures at the school on  
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## Basic objects in Operator Algebras.

- Let  $\mathcal{H}$  be a complex Hilbert space. By  $\mathcal{B}(\mathcal{H})$ , we denote the vector space of all continuous linear mappings  $T: \mathcal{H} \rightarrow \mathcal{H}$ . Recall that  $\mathcal{B}(\mathcal{H})$  is an algebra, when it is equipped with the multiplication given by composition of mappings:  $ab = a \circ b$ .
- A unital  $C^*$ -algebra is a subalgebra of  $\mathcal{B}(\mathcal{H})$ , which contains the unit  $\mathbf{1}$  of  $\mathcal{B}(\mathcal{H})$ , and which is closed w.r.t. the adjoint operation  $T \mapsto T^*$ :

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle, \quad (\xi, \eta \in \mathcal{H}),$$

and w.r.t. the operator norm:

$$\|T\| = \sup\{\|T\xi\| \mid \xi \in \mathcal{H}, \|\xi\| \leq 1\}.$$

- A *von Neumann algebra* is a unital  $C^*$ -algebra which is, in addition, closed in the weak operator topology, i.e. the weak topology on  $\mathcal{B}(\mathcal{H})$  induced by the linear functionals:

$$T \mapsto \langle T\xi, \eta \rangle, \quad (T \in \mathcal{B}(\mathcal{H}), \xi, \eta \in \mathcal{H}).$$

- A *state* on a unital  $C^*$ -algebra  $\mathcal{A}$  is a linear functional  $\phi: \mathcal{A} \rightarrow \mathbb{C}$ , which is positive (i.e.  $\phi(a^*a) \geq 0$  for any  $a$  in  $\mathcal{A}$ ) and satisfied that  $\phi(\mathbf{1}) = 1$ .
- A state  $\phi$  on a unital  $C^*$ -algebra  $\mathcal{A}$  is *faithful*, if  $\phi(a^*a) > 0$  for any non-zero operator  $a$  in  $\mathcal{A}$ .
- A state  $\phi$  on a von Neumann algebra  $\mathcal{A}$  is *normal*, if  $\phi|_{(\mathcal{A})_1}$  is continuous w.r.t. the weak operator topology.
- A state  $\tau$  on a unital  $C^*$ -algebra  $\mathcal{A}$  is a *trace*, if it satisfies that

$$\tau(ab) = \tau(ba), \quad (a, b \in \mathcal{A}).$$

## Non-commutative probability spaces.

**Definition.** A  $C^*$ -probability space is a pair  $(\mathcal{A}, \phi)$ , where  $\mathcal{A}$  is a unital  $C^*$ -algebra and  $\phi$  is a state on  $\mathcal{A}$ .

**Definition.** A  $W^*$ -probability space is a pair  $(\mathcal{A}, \tau)$ , where  $\mathcal{A}$  is a von Neumann algebra and  $\tau$  is a faithful normal state on  $\mathcal{A}$ .

**Random variables.** If  $(\mathcal{A}, \phi)$  is a  $C^*$ -probability space, we shall consider all operators in  $\mathcal{A}$  as “non-commutative random variables”. For  $a$  in  $\mathcal{A}$ , we think of  $\phi(a)$  as the *expectation of  $a$*  w.r.t.  $\phi$ .

## Spectral theory.

**Spectrum.** Let  $a$  be an operator in a unital  $C^*$ -algebra  $\mathcal{A}$ . Then the spectrum  $\text{sp}(a)$  of  $a$  is the set

$$\text{sp}(a) = \{\lambda \in \mathbb{C} \mid a - \lambda \mathbf{1} \text{ is not invertible}\}.$$

Since  $a$  is a bounded operator,  $\text{sp}(a)$  is a compact subset of  $\mathbb{C}$ .

**Spectral mapping.** If  $a$  is a selfadjoint operator in a unital  $C^*$ -algebra  $\mathcal{A}$ , then  $\text{sp}(a)$  is a compact subset of  $\mathbb{R}$ . For any polynomial  $p$  in  $\mathbb{C}[X]$ , we let  $p(a) \in \mathcal{A}$  have the obvious meaning. Then the mapping

$$p \mapsto p(a): \mathbb{C}[X] \rightarrow \mathcal{A},$$

has a unique extension to a mapping

$$\Phi_a: f \mapsto f(a): C(\text{sp}(a)) \rightarrow \mathcal{A}.$$

This mapping satisfies the properties:

$$(f + zg)(a) = f(a) + zg(a),$$

$$fg(a) = f(a)g(a),$$

$$\overline{f}(a) = f(a)^*,$$

$$\text{sp}(f(a)) = f(\text{sp}(a)),$$

for any functions  $f, g$  in  $C(\text{sp}(a))$  and any complex number  $z$ .

If  $\mathcal{A}$  is a von Neumann algebra,  $\Phi_a$  can be extended to the class of all bounded Borel functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ .

## Distributions of non-commutative random variables.

**Classical Probability.** Consider a probability space  $(\Omega, \mathcal{F}, P)$  and random variables  $X: (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B})$ . Any such random variable  $X$  gives rise to a probability measure  $\mu_X$  on  $(\mathbb{R}, \mathcal{B})$  via:

$$\mu_X(B) = P(X \in B) = P(X^{-1}(B)), \quad (B \in \mathcal{B}).$$

We call  $\mu_X$  the distribution of  $X$  and denote it also by  $L\{X\}$ . This distribution satisfies, moreover, that

$$\int_{\mathbb{R}} f(t) \mu_X(dt) = \mathbb{E}\{f(X)\},$$

for any bounded Borel function  $f: \mathbb{R} \rightarrow \mathbb{R}$ .

**Non-commutative Probability.** Let  $(\mathcal{A}, \phi)$  be a  $C^*$ -probability space, and let  $a$  be a selfadjoint operator in  $\mathcal{A}$  (i.e.  $a^* = a$ ). Then there exists a unique probability measure  $\mu_a$  on  $(\mathbb{R}, \mathcal{B})$ , such that

$$\int_{\mathbb{R}} f(t) \mu_a(dt) = \phi(f(a)), \quad (f \in C(\text{sp}(a))).$$

We call  $\mu_a$  the distribution of  $a$  w.r.t.  $\phi$ , and denote it also by  $L\{a\}$ . We always have that  $\text{supp}(\mu_a) \subseteq \text{sp}(a)$ , so, in particular,  $\mu_a$  is compactly supported.

**Classical Independence.** Two random variables  $X$  and  $Y$  on  $(\Omega, \mathcal{F}, P)$  are independent if and only if

$$\mathbb{E}\{[f(X) - \mathbb{E}\{f(X)\}] \cdot [g(Y) - \mathbb{E}\{g(Y)\}]\} = 0,$$

for any bounded Borel functions  $f, g: \mathbb{R} \rightarrow \mathbb{R}$ .

**Free Independence.** Let  $a_1, a_2, \dots, a_r$  be operators in a  $C^*$ -algebra  $\mathcal{A}$ , and let  $\phi: \mathcal{A} \rightarrow \mathbb{C}$  be a state. Then  $a_1, a_2, \dots, a_r$  are *freely independent* w.r.t.  $\phi$  if

$$\phi\{[p_1(a_{i_1}) - \phi(p_1(a_{i_1}))] \cdots [p_k(a_{i_k}) - \phi(p_k(a_{i_k}))]\} = 0,$$

for all  $k$  in  $\mathbb{N}$ , all polynomials  $p_1, \dots, p_k$  in  $\mathbb{C}[X]$  and all indices  $i_1, i_2, \dots, i_k$  in  $\{1, 2, \dots, r\}$ , such that

$$i_1 \neq i_2, i_2 \neq i_3, \dots, i_{k-1} \neq i_k.$$

**Remark.** If  $a_1, \dots, a_r$  are freely independent operators in a  $C^*$ -probability space  $(\mathcal{A}, \phi)$ , then all mixed moments:

$$\phi(a_{i_1} a_{i_2} \cdots a_{i_p}), \quad (p \in \mathbb{N}, i_1, i_2, \dots, i_p \in \{1, 2, \dots, r\})$$

are uniquely determined by the marginal distributions

$$L\{a_1\}, \dots, L\{a_r\}.$$

**Example. [Non-trivial classical random variables cannot be freely independent]**

Let  $X$  and  $Y$  be random variables on  $(\Omega, \mathcal{F}, P)$  with moments of all orders (so that the definition of  $X$  being free from  $Y$  makes sense!!). If  $X$  and  $Y$  are freely independent w.r.t.  $\mathbb{E}$ , then one of them has to be a constant (almost surely). Indeed, assume that  $X$  and  $Y$  are free and put  $m_X = \mathbb{E}(X)$  and  $m_Y = \mathbb{E}(Y)$ . Then by free independence we have:

$$\begin{aligned} 0 &= \mathbb{E}\{(X - m_X)(Y - m_Y)(X - m_X)(Y - m_Y)\} \\ &= \mathbb{E}\{(X - m_X)^2(Y - m_Y)^2\} \\ &= \mathbb{E}\{(X - m_X)^2\}\mathbb{E}\{(Y - m_Y)^2\}, \end{aligned}$$

where the last equality is another consequence of the free independence. Hence, either  $X$  or  $Y$  has to be a constant (almost surely).

## The Central Limit Theorem.

**Theorem [Classical CLT].** Let  $X_1, X_2, X_3, \dots$ , be a sequence of i.i.d. (classical) random variables defined on the same probability space  $(\Omega, \mathcal{F}, P)$ , and such that  $\mathbb{E}\{X_1^2\} < \infty$ . Put  $\xi = \mathbb{E}\{X_1\}$  and  $\sigma^2 = \mathbb{V}\{X_1\}$ , and consider the normalized random variables:

$$S_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n \left( \frac{X_j - \xi}{\sigma} \right), \quad (n \in \mathbb{N}).$$

Then

$$S_n \xrightarrow{d} N(0, 1), \quad \text{as } n \rightarrow \infty.$$

**Theorem [Voiculescu].** Let  $(a_j)_{j \in \mathbb{N}}$  be a family of freely independent random variables in a  $C^*$ -probability space  $(\mathcal{A}, \phi)$ , and such that

- (a)  $\phi(a_j) = 0$ , for all  $j$  in  $\mathbb{N}$ .
- (b)  $\sup_{j \in \mathbb{N}} |\phi(a_j^k)| < \infty$ , for all  $k$  in  $\{2, 3, \dots\}$ .
- (c)  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \phi(a_j^2) = 1$ .

Consider the normalized sums:

$$s_n = \frac{1}{\sqrt{n}}(a_1 + \dots + a_n), \quad (n \in \mathbb{N}).$$

Then the sequence  $s_n$  converges, in moments, towards the semi-circle law  $\frac{1}{2\pi} \sqrt{4 - t^2} \cdot 1_{[-2, 2]}(t) dt$ , i.e.

$$\lim_{n \rightarrow \infty} \phi(s_n^k) = \frac{1}{2\pi} \int_{-2}^2 t^k \sqrt{4 - t^2} dt, \quad (k \in \mathbb{N}).$$

**Definition.** Let  $(\mathcal{A}, \phi)$  be a  $C^*$ -probability space. A free semi-circular system in  $(\mathcal{A}, \phi)$  is a family  $\{x_1, x_2, \dots, x_r\}$  of selfadjoint operators in  $(\mathcal{A}, \phi)$  satisfying that

- $x_1, \dots, x_r$  are freely independent.
- For each  $i$  in  $\{1, 2, \dots, r\}$ ,  $L\{x_i\} = \frac{1}{2\pi} \sqrt{4 - t^2} \cdot 1_{[-2,2]}(t) dt$ .

## Free group factors.

Let  $G$  be a discrete group and consider the Hilbert space

$$\ell^2(G) = \{(y_g)_{g \in G} \mid y_g \in \mathbb{C}, \sum_{g \in G} |y_g|^2 < \infty\}.$$

For each  $g$  in  $G$ , let  $L_g: \ell^2(G) \rightarrow \ell^2(G)$  be the linear operator in  $\mathcal{B}(\ell^2(G))$ , given by:

$$L_g((y_h)_{h \in G}) = (y_{g^{-1}h})_{h \in G}.$$

Then  $L_g$  is a unitary in  $\mathcal{B}(\ell^2(G))$ , and we put

- $\mathcal{L}(G) = W^*(\{L_g \mid g \in G\}) \subseteq \mathcal{B}(\ell^2(G))$ .
- $C_{\text{red}}^*(G) = C^*(\{L_g \mid g \in G\}) \subseteq \mathcal{B}(\ell^2(G))$ .

The von Neumann algebra  $\mathcal{L}(G)$  is called the group von Neumann algebra associated to  $G$ . The  $C^*$ -algebra  $C_{\text{red}}^*(G)$  is called the reduced  $C^*$ -algebra associated to  $G$ .

**Definition.** A group  $G$  is called an *i.c.c. group*, if, for any  $h$  in  $G \setminus \{1_G\}$ , the conjugacy class  $\{ghg^{-1} \mid g \in G\}$  is infinite.

**Theorem [Murray-von Neumann].** If  $G$  is an i.c.c. group, then  $\mathcal{L}(G)$  is a factor, i.e.

$$\{a \in \mathcal{L}(G) \mid \forall b \in \mathcal{L}(G): ab = ba\} = \mathbb{C}\mathbf{1}.$$

Furthermore,  $\mathcal{L}(G)$  is infinite-dimensional and has a faithful normal trace  $\tau$ . Altogether,  $\mathcal{L}(G)$  is a von Neumann factor of type  $\text{II}_1$ .

**Question I.** Suppose  $G$  and  $H$  are two i.c.c. groups. Does the following implication hold:

$$G \not\cong H \implies \mathcal{L}(G) \not\cong \mathcal{L}(H)?$$

**Answer:** No: If  $G$  is i.c.c. and amenable then  $\mathcal{L}(G)$  is isomorphic to the hyperfinite  $\text{II}_1$ -factor. Two non-isomorphic, amenable, i.c.c. groups are:

$$G = S_\infty \quad \text{and} \quad H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Q}_+, b \in \mathbb{Q} \right\}.$$

**Question II.** For each  $n$  in  $\mathbb{N}$ , let  $\mathbb{F}_n$  denote the free group on  $n$  generators. If  $n, m \in \mathbb{N}$ ,  $n \neq m$ , then  $\mathbb{F}_n \not\cong \mathbb{F}_m$ . But is it true that

$$\mathcal{L}(\mathbb{F}_n) \simeq \mathcal{L}(\mathbb{F}_m)?$$

**Theorem [Voiculescu].** Let  $n$  be a positive integer. Then the  $\text{II}_1$ -factor  $\mathcal{L}(\mathbb{F}_n)$  is generated, as a von Neumann algebra, by a semi-circular system

$$\{x_1, x_2, \dots, x_n\} \subseteq \mathcal{L}(\mathbb{F}_n).$$

**Theorem [Voiculescu, Dykema, Radulescu].** One of the following two extremes hold:

- (i) For any  $n, m$  in  $\mathbb{N}$ ,  $\mathcal{L}(\mathbb{F}_n) \simeq \mathcal{L}(\mathbb{F}_m)$ .
- (ii) For any distinct  $n, m$  in  $\mathbb{N}$ ,  $\mathcal{L}(\mathbb{F}_n) \not\cong \mathcal{L}(\mathbb{F}_m)$ .

## Creation Operators on full Fock space.

Let  $\mathcal{H}$  be a (complex) Hilbert space with orthonormal basis  $(e_i)_{i \in I}$ , and consider the full Fock space

$$\mathcal{T}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n \in \mathbb{N}} \mathcal{H}^{\otimes n}$$

Then with

$$E = \{e_{i_1} \otimes \cdots \otimes e_{i_n} \mid n \in \mathbb{N}, i_1, \dots, i_n \in I\},$$

the set  $\{\Omega\} \cup E$  forms an orthonormal basis for  $\mathcal{T}(\mathcal{H})$ .

For  $h$  in  $\mathcal{H}$  the *left creation operator*  $\ell(h): \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$  is given by:

$$\ell(h)\Omega = h,$$

$$\ell(h)h_1 \otimes \cdots \otimes h_n = h \otimes h_1 \otimes \cdots \otimes h_n, \quad (h_1, \dots, h_n \in \mathcal{H}).$$

It is routine to verify that

$$\ell(h)^*\Omega = 0,$$

$$\ell(h)^*h_1 \otimes \cdots \otimes h_n = \begin{cases} \langle h, h_1 \rangle h_2 \otimes \cdots \otimes h_n, & \text{if } n \geq 2 \\ \langle h, h_1 \rangle \Omega, & \text{if } n = 1. \end{cases}$$

**Theorem [Voiculescu].** Let  $(e_i)_{i \in I}$  be an orthonormal basis for  $\mathcal{H}$ , and for each  $i$  in  $I$  put:

$$s(e_i) = \ell(e_i) + \ell(e_i)^*.$$

Then the operators  $s(e_i)$ ,  $i \in I$ , are freely independent, and for each  $i$  in  $I$ ,

$$L\{s(e_i)\} = \frac{1}{2\pi} \sqrt{4 - t^2} \cdot 1_{[-2,2]}(t) dt.$$

## Classical and free convolution.

Let  $\mu_1$  and  $\mu_2$  be probability measures on  $\mathbb{R}$ .

**Classical convolution.** Consider *independent* (classical) random variables  $X_1$  and  $X_2$ , such that  $L\{X_1\} = \mu_1$  and  $L\{X_2\} = \mu_2$ . Then

$$\mu_1 * \mu_2 := L\{X_1 + X_2\}$$

**Free (additive) convolution.** Consider *freely independent* selfadjoint operators  $a_1$  and  $a_2$ , such that  $L\{a_1\} = \mu_1$  and  $L\{a_2\} = \mu_2$ . Then

$$\mu_1 \boxplus \mu_2 := L\{a_1 + a_2\}.$$

## Unbounded distributions/operators.

Assume that  $(\mathcal{A}, \tau)$  is a  $W^*$ -probability space, such that  $\mathcal{A}$  is acting on the Hilbert space  $\mathcal{H}$ . If  $a$  is a (possibly unbounded) selfadjoint operator in  $\mathcal{H}$ , then  $a$  is *affiliated with  $\mathcal{A}$* , if  $f(a) \in \mathcal{A}$  for any bounded Borel function  $f: \mathbb{R} \rightarrow \mathbb{R}$ .

### Facts:

- If  $a$  is a selfadjoint operator affiliated with  $\mathcal{A}$ , there exists, as in the bounded case, a unique probability measure  $L\{a\}$  on  $\mathbb{R}$ , such that

$$\int_{\mathbb{R}} f(t) L\{a\}(dt) = \tau(f(a)), \quad (f \in \mathcal{B}_b(\mathbb{R})).$$

We refer to  $L\{a\}$  as the (spectral) distribution of  $a$ .

- If  $a_1, a_2, \dots, a_r$  are selfadjoint operators affiliated with  $(\mathcal{A}, \tau)$ , we say that  $a_1, a_2, \dots, a_r$  are freely independent, if, for any  $f_1, f_2, \dots, f_r \in \mathcal{B}_b(\mathbb{R})$ ,  $f_1(a_1), f_2(a_2), \dots, f_r(a_r)$  are freely independent.
- Any two probability measures  $\mu_1$  and  $\mu_2$  on  $\mathbb{R}$  can be realized as the distributions of two freely independent selfadjoint operators  $a_1$  and  $a_2$  affiliated with some  $W^*$ -probability space  $(\mathcal{A}, \tau)$ .
- If  $a_1$  and  $a_2$  are freely independent selfadjoint operators affiliated with  $\mathcal{A}$ , then  $L\{a_1 + a_2\}$  is uniquely determined by  $L\{a_1\}$  and  $L\{a_2\}$ .

## Cumulant transforms.

**Classical cumulant transform:** For a probability measure  $\mu$  on  $\mathbb{R}$ , we define the (classical) cumulant transform  $C_\mu$  by

$$C_\mu(u) = \log \int_{\mathbb{R}} e^{isu} \mu(ds), \quad (u \in \mathbb{R}).$$

The key property of the cumulant transform is that

$$C_{\mu_1 * \mu_2}(u) = C_{\mu_1}(u) + C_{\mu_2}(u), \quad (u \in \mathbb{R}),$$

for any probability measures  $\mu_1, \mu_2$  on  $\mathbb{R}$ .

**Free cumulant transform:** For a probability measure  $\mu$  on  $\mathbb{R}$ , the *Cauchy transform*  $G_\mu: \mathbb{C}^+ \rightarrow \mathbb{C}^-$  is defined by:

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t} \mu(dt), \quad (z \in \mathbb{C}^+).$$

It turns out that the mapping  $F_\mu := \frac{1}{G_\mu}: \mathbb{C}^+ \rightarrow \mathbb{C}^+$  always has a right inverse,  $F_\mu^{-1}$ , defined on a region of the form:

$$\Gamma(\eta, M) = \{x + iy \in \mathbb{C}^+ \mid x^2 + y^2 > M^2, |x| < \eta y\}$$

where  $\eta$  and  $M$  are positive numbers.

The free cumulant transform  $\mathcal{C}_\mu$  is defined by

$$\mathcal{C}_\mu(z) = zF_\mu^{-1}\left(\frac{1}{z}\right) - 1 = zG_\mu^{-1}(z) - 1 = zR_\mu(z),$$

for  $z$  in the region  $\{x - iy \in \mathbb{C}^- \mid x^2 + y^2 < M^{-2}, |x| < \eta y\}$ .

The key property of the free cumulant transform is that

$$\mathcal{C}_{\mu_1 \boxplus \mu_2}(z) = \mathcal{C}_{\mu_1}(z) + \mathcal{C}_{\mu_2}(z),$$

for any probability measures  $\mu_1, \mu_2$  on  $\mathbb{R}$ .

**Definition.** A probability measure  $\mu$  on  $\mathbb{R}$  is  $\boxplus$ -infinitely divisible if, for any  $n$  in  $\mathbb{N}$ , there exists a probability measure  $\mu_n$  on  $\mathbb{R}$ , such that

$$\mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ terms}}.$$

We denote by  $\mathcal{JD}(\boxplus)$  the class of all  $\boxplus$ -infinitely divisible probability measures on  $\mathbb{R}$ .

Similarly,  $\mathcal{JD}(\ast)$  denotes the class of all classically infinitely divisible probability measures on  $\mathbb{R}$ .

## The Bercovici-Pata bijection.

**Theorem. [Lévy-Khintchine]** Let  $\mu$  be a probability measure on  $\mathbb{R}$  with cumulant function  $C_\mu$ . Then  $\mu$  is  $*$ -infinitely divisible, if and only if  $C_\mu$  has the Lévy-Khintchine representation:

$$C_\mu(u) = i\eta u - \frac{1}{2}au^2 + \int_{\mathbb{R}} (e^{iut} - 1 - iut1_{[-1,1]}(t)) \rho(dt), \quad (u \in \mathbb{R}),$$

where  $\eta \in \mathbb{R}$ ,  $a \geq 0$  and  $\rho$  is a Lévy measure on  $\mathbb{R}$ , i.e.,

$$\rho(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min\{1, x^2\} \rho(dx) < \infty.$$

The triplet  $(a, \rho, \eta)$  is uniquely determined, and it is called the *generating triplet* for  $\mu$ .

**Theorem [Bercovici & Voiculescu].** Let  $\nu$  be a probability measure on  $\mathbb{R}$  with free cumulant transform  $\mathcal{C}_\nu$ . Then  $\nu$  is  $\boxplus$ -infinitely divisible if and only if there exist a non-negative number  $a$ , a real number  $\eta$  and a Lévy measure  $\rho$ , such that

$$\mathcal{C}_\nu(z) = \eta z + az^2 + \int_{\mathbb{R}} \left( \frac{1}{1-tz} - 1 - tz1_{[-1,1]}(t) \right) \rho(dt), \quad (z \in \mathbb{C}^-).$$

In that case, the triplet  $(a, \rho, \eta)$  is uniquely determined and is called the *free generating triplet* for  $\nu$ .

**Definition.** The Bercovici-Pata bijection  $\Lambda: \mathcal{JD}(\ast) \rightarrow \mathcal{JD}(\boxplus)$  is defined as follows: For a measure  $\mu$  in  $\mathcal{JD}(\ast)$  with generating triplet  $(a, \rho, \eta)$ ,  $\Lambda(\mu)$  is the measure in  $\mathcal{JD}(\boxplus)$  with *free* generating triplet  $(a, \rho, \eta)$ .

**Theorem [Barndorff-Nielsen+T].** For any probability measure  $\mu$  in  $\mathcal{JD}(\ast)$ , we have

$$\mathcal{C}_{\Lambda(\mu)}(i\zeta) = \int_{\mathbb{R}} C_{\mu}(\zeta x) e^{-x} dx, \quad (\zeta < 0).$$

**Proof.** Recall that for any  $z$  in  $\mathbb{C}$  with  $\operatorname{Re} z < 1$  we have

$$\frac{1}{1-z} = \int_0^{\infty} e^{zx} e^{-x} dx$$

implying that for  $\zeta$  real with  $\zeta \leq 0$

$$\frac{1}{1-i\zeta t} - 1 - i\zeta t 1_{[-1,1]}(t) = \int_0^{\infty} (e^{i\zeta tx} - 1 - i\zeta tx 1_{[-1,1]}(t)) e^{-x} dx.$$

Now, let  $\mu$  from  $\mathcal{JD}(\ast)$  be given and let  $(a, \rho, \eta)$  be the generating triplet for  $\mu$ . Then for  $\zeta$  in  $] -\infty, 0[$  we have

$$\begin{aligned} & \mathcal{C}_{\Lambda(\mu)}(i\zeta) \\ &= -a\zeta^2 + i\eta\zeta + \int_{\mathbb{R}} \left( \frac{1}{1-i\zeta t} - 1 - i\zeta t 1_{[-1,1]}(t) \right) \rho(dt) \\ &= -a\zeta^2 + i\eta\zeta + \int_{\mathbb{R}} \int_0^{\infty} (e^{i\zeta tx} - 1 - i\zeta tx 1_{[-1,1]}(t)) e^{-x} dx \rho(dt) \\ &= -a\zeta^2 + i\eta\zeta + \int_0^{\infty} e^{-x} \int_{\mathbb{R}} (e^{i\zeta tx} - 1 - i\zeta tx 1_{[-1,1]}(t)) \rho(dt) dx \\ &= \int_0^{\infty} e^{-x} \left( -\frac{1}{2}a\zeta^2 x^2 + i\eta\zeta x + \int_{\mathbb{R}} (e^{i\zeta tx} - 1 - i\zeta tx 1_{[-1,1]}(t)) \rho(dt) \right) dx, \\ &= \int_0^{\infty} e^{-x} C_{\mu}(\zeta x) dx, \end{aligned}$$

where we have used the formula:  $n! = \int_0^{\infty} x^n e^{-x} dx$ , ( $n \in \mathbb{N}$ ).

**Notation.** Let  $\mu$  be a probability measure on  $\mathbb{R}$  and let  $X$  be a random variable such that  $L\{X\} = \mu$ . Then for any real constant  $c$  we put  $D_c\mu = L\{cX\}$ .

**Theorem [Barndorff-Nielsen+T].** The Bercovici-Pata bijection  $\Lambda$  has the following algebraic properties:

- (i) If  $\mu_1, \mu_2 \in \mathcal{JD}(\ast)$ , then  $\Lambda(\mu_1 \ast \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$ .
- (ii) If  $\mu \in \mathcal{JD}(\ast)$  and  $c \in \mathbb{R}$ , then  $\Lambda(D_c\mu) = D_c\Lambda(\mu)$ .
- (iii) For any  $c$  in  $\mathbb{R}$ ,  $\Lambda(\delta_c) = \delta_c$ .

**Proof of (ii):** For  $\zeta < 0$  we have

$$\begin{aligned} \mathfrak{C}_{\Lambda(D_c\mu)}(i\zeta) &= \int_{\mathbb{R}} C_{D_c\mu}(\zeta x) e^{-x} dx = \int_{\mathbb{R}} C_{\mu}(c\zeta x) e^{-x} dx \\ &= \mathfrak{C}_{\Lambda(\mu)}(ic\zeta) = \mathfrak{C}_{D_c\Lambda(\mu)}(i\zeta). \end{aligned}$$

The result then follows from uniqueness of analytic continuation.

## Examples.

(1) Let  $\mu$  be the standard Gaussian distribution, i.e.

$$\mu(dx) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2) dx.$$

Then  $\Lambda(\mu)$  is the semi-circle distribution, i.e.,

$$\Lambda(\mu)(dx) = \frac{1}{2\pi} \sqrt{4 - x^2} \cdot 1_{[-2,2]}(x) dx.$$

(2) Let  $\mu$  be the Poisson distribution with parameter  $\lambda > 0$ , i.e.

$$\mu(\{n\}) = e^{-\lambda} \frac{\lambda^n}{n!}, \quad (n \in \mathbb{N}_0).$$

Then  $\Lambda(\mu)$  is the free Poisson distribution, i.e.

$$\begin{aligned} & \Lambda(\mu)(dx) \\ &= \begin{cases} (1 - \lambda)\delta_0 + \frac{1}{2\pi x} \sqrt{(x - a)(b - x)} \cdot 1_{[a,b]}(x) dx, & \text{if } 0 \leq \lambda \leq 1, \\ \frac{1}{2\pi x} \sqrt{(x - a)(b - x)} \cdot 1_{[a,b]}(x) dx, & \text{if } \lambda > 1, \end{cases} \end{aligned}$$

where  $a = (1 - \sqrt{\lambda})^2$  and  $b = (1 + \sqrt{\lambda})^2$ .

(3) Let  $\mu$  be the Cauchy distribution, i.e.

$$\mu(dx) = \frac{1}{\pi(1 + x^2)} dx.$$

Then  $\Lambda(\mu) = \mu$ .

## Classes of probability measures on $\mathbb{R}$ .

In classical probability theory we have the following hierarchy:

$$\mathcal{G}(\ast) \subset \mathcal{S}(\ast) \subset \mathcal{L}(\ast) \subset \mathcal{ID}(\ast) \subset \mathcal{P}$$

where

- $\mathcal{P}$  is the class of all probability measures on  $\mathbb{R}$ .
- $\mathcal{ID}(\ast)$  is the class of infinitely divisible probability measures on  $\mathbb{R}$ , i.e.

$$\mu \in \mathcal{ID}(\ast) \iff \forall n \in \mathbb{N} \exists \mu_n \in \mathcal{P}: \mu = \underbrace{\mu_n \ast \mu_n \ast \cdots \ast \mu_n}_{n \text{ terms}}.$$

- $\mathcal{L}(\ast)$  is the class of selfdecomposable probability measures on  $\mathbb{R}$ , i.e.

$$\mu \in \mathcal{L}(\ast) \iff \forall c \in ]0, 1[ \exists \mu_c \in \mathcal{P}: \mu = D_c \mu \ast \mu_c,$$

where  $D_c \mu(B) = \mu(c^{-1}B)$  for any Borel set  $B$  in  $\mathbb{R}$ .

- $\mathcal{S}(\ast)$  is the class of stable probability measures on  $\mathbb{R}$ , i.e.

$$\mu \in \mathcal{S}(\ast) \iff \{\psi(\mu) \mid \psi: \mathbb{R} \rightarrow \mathbb{R}, \text{ increasing affine transformation}\} \\ \text{is stable w.r.t. } \ast.$$

- $\mathcal{G}(\ast)$  is the class of Gaussian (or normal) distributions on  $\mathbb{R}$ .

## Corresponding classes in free probability.

- $\mathcal{JD}(\boxplus)$  is the class of  $\boxplus$ -infinitely divisible probability measures on  $\mathbb{R}$ , i.e.

$$\mu \in \mathcal{JD}(\boxplus) \iff \forall n \in \mathbb{N} \exists \mu_n \in \mathcal{P}: \mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ terms}}.$$

- $\mathcal{L}(\boxplus)$  is the class of  $\boxplus$ -selfdecomposable probability measures on  $\mathbb{R}$ , i.e.

$$\mu \in \mathcal{L}(\boxplus) \iff \forall c \in ]0, 1[ \exists \mu_c \in \mathcal{P}: \mu = D_c \mu \boxplus \mu_c,$$

where  $D_c \mu(B) = \mu(c^{-1}B)$  for any Borel set  $B$  in  $\mathbb{R}$ .

- $\mathcal{S}(\boxplus)$  is the class of  $\boxplus$ -stable probability measures on  $\mathbb{R}$ , i.e.

$$\mu \in \mathcal{S}(\boxplus) \iff \{\psi(\mu) \mid \psi: \mathbb{R} \rightarrow \mathbb{R}, \text{ increasing affine transformation}\} \\ \text{is stable w.r.t. } \boxplus.$$

- $\mathcal{G}(\boxplus)$  is the class of semi-circle distributions on  $\mathbb{R}$ .

**Question:** Do we have the same hierarchy in free probability:

$$\mathcal{G}(\boxplus) \subset \mathcal{S}(\boxplus) \subset \mathcal{L}(\boxplus) \subset \mathcal{JD}(\boxplus)?$$

## Remarks.

- It is well-known that  $\mathcal{G}(\boxplus) \subset \mathcal{S}(\boxplus)$ .
- Using (for example) the free cumulant transform, it is not hard to show that  $\mathcal{S}(\boxplus) \subset \mathcal{L}(\boxplus)$ .
- It is well-known that  $\mathcal{S}(\boxplus) \subset \mathcal{JD}(\boxplus)$  [Bercovici & Voiculescu].

**Proof of:**  $\mathcal{L}(\boxplus) \subset \mathcal{ID}(\boxplus)$ .

**Theorem. [Bercovici & Pata]** For each  $n$  in  $\mathbb{N}$ , let  $\mu_{1,n}, \mu_{2,n}, \dots, \mu_{n,n}$  be probability measures on  $\mathbb{R}$ , and put:

$$\mu_n = \mu_{n,1} \boxplus \mu_{n,2} \boxplus \dots \boxplus \mu_{n,n}.$$

Assume that

$$\forall \epsilon > 0: \lim_{n \rightarrow \infty} \max_{j=1,2,\dots,n} \mu_{j,n}(\mathbb{R} \setminus [-\epsilon, \epsilon]) = 0. \quad (*)$$

Then, if  $\mu_n \xrightarrow{w} \mu$  as  $n \rightarrow \infty$ , for some probability measure  $\mu$ , this measure  $\mu$  has to be  $\boxplus$ -infinitely divisible.

Now, if  $\mu \in \mathcal{L}(\boxplus)$ , we have for any  $c$  in  $[0, 1]$ ,

$$\begin{aligned} \mu &= D_c \mu \boxplus \mu_c \\ &= D_c(D_c \mu \boxplus \mu_c) \boxplus \mu_c \\ &= D_{c^2} \mu \boxplus D_c \mu_c \boxplus \mu_c \\ &= \dots \\ &= D_{c^n} \mu \boxplus D_{c^{n-1}} \mu_c \boxplus D_{c^{n-2}} \mu_c \boxplus \dots \boxplus D_c \mu_c \boxplus \mu_c. \end{aligned}$$

Now choose  $c = c_n$ , so that the family

$$\{D_{c^n} \mu, D_{c^{n-1}} \mu_c, D_{c^{n-2}} \mu_c, \dots, D_c \mu_c, \mu_c\}$$

satisfies condition  $(*)$  above.

**Theorem.** [Bercovici & Pata] The Bercovici-Pata bijection preserves stability, i.e.  $\Lambda(\mathcal{S}(*)) = \mathcal{S}(\boxplus)$ .

**Proposition** [Barndorff-Nielsen+T]. The Bercovici-Pata bijection  $\Lambda$  preserves selfdecomposability, i.e.  $\Lambda(\mathcal{L}(*)) = \mathcal{L}(\boxplus)$ .

**Proof.** Suppose  $\mu \in \mathcal{L}(*)$  and  $c \in ]0, 1[$ . Then  $\mu = D_c\mu * \mu_c$ , where, by necessity,  $\mu_c \in \mathcal{J}\mathcal{D}(*)$  too (classical result). Hence, by the algebraic properties of  $\Lambda$ ,

$$\Lambda(\mu) = \Lambda(D_c\mu * \mu_c) = D_c\Lambda(\mu) \boxplus \Lambda(\mu_c).$$

**A significant difference between the classical and free theories.**

**Cramér's Theorem.** Suppose that  $\mu_1$  and  $\mu_2$  are probability measures on  $\mathbb{R}$ , such that

$$\mu_1 * \mu_2 = N(\xi, \sigma^2),$$

for some  $\xi$  in  $\mathbb{R}$  and  $\sigma^2$  in  $]0, \infty[$ . Then  $\mu_1$  and  $\mu_2$  themselves have to be Gaussian.

**Theorem [Bercovici-Voiculescu].** There exist non semi-circular distributions  $\mu_1$  and  $\mu_2$  on  $\mathbb{R}$ , such that

$$\mu_1 \boxplus \mu_2 = \frac{1}{2\pi} \sqrt{4 - t^2} \cdot 1_{[-2,2]}(t) dt.$$

**Theorem [Barndorff-Nielsen+T].** The Percovici-Pata bijection is a homeomorphism w.r.t. weak convergence, i.e. for measures  $\mu, \mu_1, \mu_2, \mu_3, \dots$  in  $\mathcal{JD}(\ast)$ , we have

$$\mu_n \xrightarrow{w} \mu \iff \Lambda(\mu_n) \xrightarrow{w} \Lambda(\mu).$$

**Idea of proof.** Suppose  $\mu_n$  (respectively  $\mu$ ) has generating triplet  $(a_n, \rho_n, \eta_n)$  (respectively  $(a, \rho, \eta)$ ). Then  $\Lambda(\mu_n)$  (respectively  $\Lambda(\mu)$ ) has free generating triplet  $(a_n, \rho_n, \eta_n)$  (respectively  $(a, \rho, \eta)$ ). Assume, for simplicity, that  $\rho_n$  and  $\rho$  have no atoms at  $-1$  or  $1$ , and consider the finite measures:

$$\begin{aligned} \sigma_n(dt) &= a_n \delta_0(dt) + \frac{1}{1+t^2} \rho_n(dt), & (n \in \mathbb{N}) \\ \sigma(dt) &= a \delta_0(dt) + \frac{1}{1+t^2} \rho(dt). \end{aligned}$$

Now

$$\mu_n \xrightarrow{w} \mu \xLeftrightarrow{\text{Gnedenko}} \eta_n \rightarrow \eta \text{ and } \sigma_n \xrightarrow{w} \sigma \iff \Lambda(\mu_n) \xrightarrow{w} \Lambda(\mu).$$

## Lévy Processes in Free Probability.

**Definition.** A free Lévy process (in law) affiliated with  $(\mathcal{A}, \tau)$  is a family  $(Z_t)_{t \geq 0}$  of selfadjoint operators affiliated with  $\mathcal{A}$ , which satisfies the following conditions:

(i) whenever  $n \in \mathbb{N}$  and  $0 \leq t_0 < t_1 < \dots < t_n$ , the increments

$$Z_{t_0}, Z_{t_1} - Z_{t_0}, Z_{t_2} - Z_{t_1}, \dots, Z_{t_n} - Z_{t_{n-1}},$$

are freely independent selfadjoint operators affiliated with  $\mathcal{A}$ .

(ii)  $Z_0 = 0$ .

(iii) for any  $s, t$  in  $[0, \infty[$ ,  $L\{Z_{s+t} - Z_s\}$  does not depend on  $s$ .

(iv) for any  $s$  in  $[0, \infty[$ ,  $L\{Z_{s+t} - Z_s\} \xrightarrow{w} \delta_0$ , as  $t \rightarrow 0$ .

**Remark.** If  $(Z_t)$  is a free Lévy process, then, as in the classical case,  $L\{Z_t\} \in \mathcal{JD}(\boxplus)$  for all  $t$ .

**Proposition.** Let  $(X_t)$  be a classical Lévy process (in law) defined on some probability space  $(\Omega, \mathcal{F}, P)$ . Then there exists a free Lévy process (in law)  $(Z_t)$  affiliated with some  $(\mathcal{A}, \tau)$ , such that

$$\Lambda(L\{X_t\}) = L\{Z_t\}, \quad (t \geq 0).$$

The proof relies on the properties of  $\Lambda$ , as well as on an existence result for processes with freely independent increments, which has been noted by Biane and Voiculescu.

**Example.** The *free Brownian motion* is the free Lévy process  $(W_t)_{t \geq 0}$ , which corresponds to the classical Brownian motion  $(B_t)_{t \geq 0}$  via the correspondence described in the proposition above. In particular

$$L\{W_t\} = \frac{1}{2\pi t} \sqrt{4t - s^2} \cdot 1_{[-\sqrt{4t}, \sqrt{4t}]}(s) \, ds, \quad (t \geq 0).$$

## Free Stochastic Integration.

**Theorem.** Let  $(Z_t)$  be a free Lévy process in law affiliated with  $(\mathcal{A}, \tau)$ , let  $0 \leq A < B$  and let  $f: [A, B] \rightarrow \mathbb{R}$  be a continuous function. Form Riemann sums:

$$T_n = \sum_{j=1}^n f(t_{n,j}^\#)(Z_{t_{n,j}} - Z_{t_{n,j-1}}),$$

where  $A = t_{n,0} < t_{n,1} < t_{n,2} < \dots < t_{n,n} = B$ ,  $t_{n,j}^\# \in [t_{n,j-1}, t_{n,j}]$  and  $\lim_{n \rightarrow \infty} \max(t_{n,j} - t_{n,j-1}) = 0$ .

Then there exists a selfadjoint operator  $T$  affiliated with  $\mathcal{A}$ , such that  $T_n \rightarrow T$  in probability, i.e.  $L\{T_n - T\} \xrightarrow{w} \delta_0$ , as  $n \rightarrow \infty$ .

$T$  does not depend on the choice of subdivisions, and we put

$$\int_A^B f(t) dZ_t = T.$$

**Theorem.** Let  $x$  be a selfadjoint operator affiliated with  $\mathcal{A}$ . Then  $L\{x\}$  is  $\boxplus$ -selfdecomposable if and only if there exists a free Lévy process (in law)  $(Z_t)$  (affiliated with some  $(\tilde{\mathcal{A}}, \tilde{\tau})$ ), such that

$$x \stackrel{d}{=} \int_0^\infty e^{-t} dZ_t.$$

## The Lévy-Itô decomposition.

**Theorem (Lévy-Itô).** Let  $(X_t)$  be a classical Lévy process and let  $\rho$  be the Lévy measure for  $L\{X_1\}$ . Assume, for simplicity, that  $\int_{-1}^1 |t| \rho(dt) < \infty$ .

Then  $(X_t)$  has a representation in the form:

$$X_t \stackrel{\text{as}}{=} at + bB_t + \int_{]0,t] \times \mathbb{R}} x N(ds, dx), \quad (*)$$

where

- $a \in \mathbb{R}$ ,  $b \geq 0$  and  $(B_t)$  is a Brownian motion.
- $N$  is a Poisson random measure on  $]0, \infty[ \times \mathbb{R}$  with intensity measure  $\text{Leb} \otimes \rho$ .
- the processes appearing in the right hand side of  $(*)$  are independent.

**Definition.** Let  $(\Theta, \mathcal{E}, \nu)$  be a  $\sigma$ -finite measure space. A Poisson random measure on  $\Theta$  with intensity measure  $\nu$  is a family  $\{N(E) \mid E \in \mathcal{E}\}$  of random variables (defined on some  $(\Omega, \mathcal{F}, P)$ ) with the following properties:

- (i) for all  $E$  in  $\mathcal{E}$ ,  $L\{N(E)\} = \text{Poiss}(\nu(E))$ ,
- (ii)  $E_1, \dots, E_r$  disjoint sets from  $\mathcal{E} \implies N(E_1), \dots, N(E_r)$  are independent,
- (iii) for all  $\omega$  in  $\Omega$ ,  $N(\cdot, \omega)$  is a measure  $\mathcal{E}$ .

**Theorem [Barndorff-Nielsen+T].** Let  $(Z_t)$  be a free Lévy-process on a Hilbert space  $\mathcal{H}$  and let  $\rho$  be the free Lévy measure for  $L\{Z_1\}$ . Assume, for simplicity, that  $\int_{-1}^1 |t| \rho(dt) < \infty$ .

Then  $(Z_t)$  has a representation in the form:

$$Z_t \stackrel{d}{=} at + bW_t + \int_{]0,t] \times \mathbb{R}} x M(ds, dx), \quad (**)$$

where

- $a \in \mathbb{R}$ ,  $b \geq 0$  and  $(W_t)$  is a free Brownian motion.
- $M$  is a free Poisson random measure on  $]0, \infty[ \times \mathbb{R}$  with intensity measure  $\text{Leb} \otimes \rho$ .
- the processes appearing in the right hand side of  $(**)$  are freely independent.

**Definition.** Let  $(\Theta, \mathcal{E}, \nu)$  be a  $\sigma$ -finite measure space, and put

$$\mathcal{E}_f = \{E \in \mathcal{E} \mid \nu(E) < \infty\}.$$

A *free* Poisson random measure on  $\Theta$  with intensity measure  $\nu$  is a family  $\{M(E) \mid E \in \mathcal{E}_f\}$  of selfadjoint operators (on some Hilbert space  $\mathcal{H}$ ) with the following properties:

- (i) for all  $E$  in  $\mathcal{E}_f$ ,  $L\{M(E)\} = \text{FreePoiss}(\nu(E))$ ,
- (ii)  $E_1, \dots, E_r$  disjoint sets from  $\mathcal{E}_f \implies M(E_1), \dots, M(E_r)$  are freely independent,
- (iii)  $E_1, \dots, E_r$  disjoint sets from  $\mathcal{E}_f \implies M(E_1 \cup \dots \cup E_r) = M(E_1) + \dots + M(E_r)$ .

## Random matrices and free probability.

**Definition.** Let  $(\Omega, \mathcal{F}, P)$  be a probability space. By  $\text{SGRM}(n, \sigma^2)$  we denote the set of random  $n \times n$  matrices  $X = (x_{ij})_{1 \leq i, j \leq n}$ , defined on  $(\Omega, \mathcal{F}, P)$ , which satisfies the following conditions:

- $\forall i \geq j: x_{ij} = \overline{x_{ji}}$ .
- the random variables  $x_{ij}$ ,  $1 \leq i \leq j \leq n$ , are independent.
- $\forall i < j: \text{Re}(x_{ij}), \text{Im}(x_{ij}) \sim \text{i.i.d. } N(0, \frac{1}{2}\sigma^2)$ .
- $\forall i: x_{ii} \sim N(0, \sigma^2)$ .

**Theorem [Wigner].** For each  $n$  in  $\mathbb{N}$ , let  $X_n$  be a random matrix in  $\text{SGRM}(n, \frac{1}{n})$ . Then for any  $p$  in  $\mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} \mathbb{E} \circ \text{tr}_n [X_n^p] = \frac{1}{2\pi} \int_{-2}^2 t^p \sqrt{4 - t^2} dt.$$

**Theorem [Voiculescu].** For each  $n \in \mathbb{N}$ , let  $X_1^{(n)}, \dots, X_r^{(n)}$  be independent random matrices from the class  $\text{SGRM}(n, \frac{1}{n})$ , and let  $(x_1, \dots, x_r)$  be a semi-circular system in a  $C^*$ -probability space  $(\mathcal{A}, \tau)$ .

Then for any polynomial  $p$  in  $r$  non-commuting variables, we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \circ \text{tr}_n [p(X_1^{(n)}, \dots, X_r^{(n)})] = \tau [p(x_1, \dots, x_r)].$$

**Theorem [T].** For each  $n$  in  $\mathbb{N}$ , let  $X_1^{(n)}, \dots, X_r^{(n)}$  be independent random matrices in  $\text{SGRM}(n, \frac{1}{n})$ , and let  $\{x_1, \dots, x_r\}$  be a semi-circular system in some  $C^*$ -probability space  $(\mathcal{A}, \tau)$ . Then for any polynomial  $p$  in  $\mathbb{C}\langle Z_1, \dots, Z_r \rangle$ , we have

$$\mathbb{E} \circ \text{tr}_n [p(X_1^{(n)}, \dots, X_r^{(n)})] = \tau [p(x_1, \dots, x_r)] + O(1/n^2),$$

as  $n \rightarrow \infty$ .

**Corollary [Hiai-Petz, T].** For each  $n$  in  $\mathbb{N}$ , let  $X_1^{(n)}, \dots, X_r^{(n)}$  be independent random matrices in  $\text{SGRM}(n, \frac{1}{n})$ , and let  $\{x_1, \dots, x_r\}$  be a semi-circular system in some  $C^*$ -probability space  $(\mathcal{A}, \tau)$ . Then for any polynomial  $p$  in  $\mathbb{C}\langle Z_1, \dots, Z_r \rangle$ , we have

$$\text{tr}_n [p(X_1^{(n)}, \dots, X_r^{(n)})] \xrightarrow{\text{a.s.}} \tau [p(x_1, \dots, x_r)],$$

as  $n \rightarrow \infty$ .

**Remark.** For  $r = 1$ , this was proved by L. Arnold in 1967.

**Theorem A [Haagerup+T]** For each  $n$  in  $\mathbb{N}$ , let  $X_1^{(n)}, \dots, X_r^{(n)}$  be independent random matrices from the class  $\text{SGRM}(n, \frac{1}{n})$ , and let  $(x_1, \dots, x_r)$  be a semi-circular system in a  $C^*$ -probability space  $(\mathcal{A}, \tau)$  with a faithful state  $\tau$ .

Then there is a  $P$ -null-set  $N \subseteq \Omega$  such that for all  $\omega \in \Omega \setminus N$  and all polynomials  $p$  in  $r$  non-commuting variables, we have

$$\lim_{n \rightarrow \infty} \|p(X_1^{(n)}(\omega), \dots, X_r^{(n)}(\omega))\| = \|p(x_1, \dots, x_r)\|.$$

**Corollary.** For any  $r$  in  $\{2, 3, 4, \dots\}$ ,  $\text{Ext}(C_{\text{red}}^*(\mathbb{F}_r))$  is *not* a group.

**The Ext semi-group.** Let  $\mathcal{A}$  be a separable, unital  $C^*$ -algebra and consider the Hilbert space  $\ell^2(\mathbb{N})$ . Consider, further, the Calkin algebra:

$$\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}).$$

Then

$$\text{Ext}(\mathcal{A}) = \{ * \text{-monomorphisms } \pi: \mathcal{A} \rightarrow \mathcal{C}(\mathcal{H}) \} / \sim$$

where

$$\pi_1 \sim \pi_2 \iff \exists u \in \mathcal{U}(\mathcal{B}(\mathcal{H})) \forall a \in \mathcal{A}: \pi_1(a) = \rho(u)\pi_2(a)\rho(u)^*.$$

$\text{Ext}(\mathcal{A})$  has a natural semi-group structure:

$$\pi_1 \oplus \pi_2(a) = \begin{pmatrix} \pi_1(a) & 0 \\ 0 & \pi_2(a) \end{pmatrix} \in \mathcal{C}(\mathcal{H}) \oplus \mathcal{C}(\mathcal{H}) \hookrightarrow \mathcal{C}(\mathcal{H} \oplus \mathcal{H}) \simeq \mathcal{C}(\mathcal{H}).$$

**Proof of**  $\liminf_{n \rightarrow \infty} \|p(X_1^{(n)}, \dots, X_r^{(n)})\| \geq \|p(x_1, \dots, x_r)\|$ .

Let  $p$  be a polynomial in  $\mathbb{C}\langle X_1, \dots, X_r \rangle$ , and put  $q = p^*p$ . Then for almost all  $\omega$  and all  $k$  in  $\mathbb{N}$ , we have by the strong version of Voiculescu's random matrix model

$$\begin{aligned} \liminf_{n \rightarrow \infty} \|p(X_1^{(n)}(\omega), \dots, X_r^{(n)}(\omega))\| & \\ & \geq \liminf_{n \rightarrow \infty} \|p(X_1^{(n)}(\omega), \dots, X_r^{(n)}(\omega))\|_{2k} \\ & = \liminf_{n \rightarrow \infty} \left( \operatorname{tr}_n [q(X_1^{(n)}(\omega), \dots, X_r^{(n)}(\omega))^k] \right)^{1/2k} \\ & = \tau[q(x_1, \dots, x_r)^k]^{1/2k} \\ & = \|p(x_1, \dots, x_r)\|_{2k}. \end{aligned}$$

Then use that

$$\lim_{k \rightarrow \infty} \|p(x_1, \dots, x_r)\|_k = \|p(x_1, \dots, x_r)\|.$$

Finally consider all polynomials in  $\mathbb{C}\langle X_1, \dots, X_r \rangle$  with coefficients in  $\mathbb{Q} + i\mathbb{Q}$ .

**Proof of**  $\limsup_{n \rightarrow \infty} \|p(X_1^{(n)}, \dots, X_r^{(n)})\| \leq \|p(x_1, \dots, x_r)\|$ .

**Theorem B (Linearization Trick).** Let  $\mathcal{A}$  and  $\mathcal{B}$  be unital  $C^*$ -algebras, and let  $x_1, \dots, x_r$  and  $y_1, \dots, y_r$  be selfadjoint operators in  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. Then the following two conditions are equivalent:

(a)  $\forall m \in \mathbb{N} \forall a_0, a_1, \dots, a_r \in M_m(\mathbb{C})_{\text{sa}}$ :

$$\text{sp}(a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r a_i \otimes x_i) \supseteq \text{sp}(a_0 \otimes \mathbf{1}_{\mathcal{B}} + \sum_{i=1}^r a_i \otimes y_i),$$

(b)  $\forall p \in \mathbb{C}\langle X_1, \dots, X_r \rangle$ :  $\|p(x_1, \dots, x_r)\| \geq \|p(y_1, \dots, y_r)\|$ .

**Theorem C (Approximate version).** Let  $x_1, \dots, x_r$  be selfadjoint operators in a  $C^*$ -algebra  $\mathcal{A}$ , and let, for each  $n$  in  $\mathbb{N}$ ,  $Z_1^{(n)}, Z_2^{(n)}, \dots, Z_r^{(n)}$  be matrices in  $M_n(\mathbb{C})_{\text{sa}}$ , such that

$$\sup_{n \in \mathbb{N}} \|Z_i^{(n)}\| < \infty, \quad (i = 1, \dots, r).$$

Assume further that

$\forall m \in \mathbb{N} \forall a_0, a_1, \dots, a_r \in M_m(\mathbb{C}) \forall \epsilon > 0$ :

$$\text{sp}\left(a_0 \otimes \mathbf{1}_n + \sum_{i=1}^r a_i \otimes Z_i^{(n)}\right) \subseteq \text{sp}\left(a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r a_i \otimes x_i\right) + ] - \epsilon, \epsilon[$$

for all  $n$  sufficiently large.

We then have

$$\forall p \in \mathbb{C}\langle X_1, \dots, X_r \rangle: \limsup_{n \rightarrow \infty} \|p(Z_1^{(n)}, \dots, Z_r^{(n)})\| \leq \|p(x_1, \dots, x_r)\|.$$

**Theorem D.** For each  $n \in \mathbb{N}$ , let  $X_1^{(n)}, \dots, X_r^{(n)}$  be independent random matrices from the class  $\text{SGRM}(n, \frac{1}{n})$ , and let  $(x_1, \dots, x_r)$  be a semi-circular system in a  $C^*$ -probability space  $(\mathcal{A}, \tau)$  with a faithful state  $\tau$ . Let, further,  $m \in \mathbb{N}$  and  $a_0, a_1, \dots, a_r$  be arbitrary matrices from  $M_m(\mathbb{C})_{\text{sa}}$ .

Then there exists a  $P$ -null-set  $N \subseteq \Omega$ , such that for all  $\omega$  in  $\Omega \setminus N$ , we have

$$\sup_{n \in \mathbb{N}} \|X_i^{(n)}(\omega)\| < \infty, \quad (i = 1, \dots, r).$$

and, for any  $\epsilon > 0$ ,

$$\text{sp}\left(a_0 \otimes \mathbf{1}_n + \sum_{i=1}^r a_i \otimes X_i^{(n)}(\omega)\right) \subseteq \text{sp}\left(a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r a_i \otimes x_i\right) + ] - \epsilon, \epsilon[ ,$$

for all  $n$  sufficiently large.

**Key-Estimates.** For each  $n \in \mathbb{N}$ , let  $X_1^{(n)}, \dots, X_r^{(n)}$  be independent random matrices from the class  $\text{SGRM}(n, \frac{1}{n})$ , and let  $(x_1, \dots, x_r)$  be a semi-circular system in a  $C^*$ -probability space  $(\mathcal{A}, \tau)$  with a faithful state  $\tau$ . Let, further,  $m \in \mathbb{N}$  and  $a_0, a_1, \dots, a_r$  be arbitrary matrices from  $M_m(\mathbb{C})_{\text{sa}}$ . Then define

$$S_n = a_0 \otimes \mathbf{1}_n + \sum_{i=1}^r a_i \otimes X_i^{(n)} \in M_{mn}(\mathbb{C})_{\text{sa}}, \quad (n \in \mathbb{N}),$$

and

$$s = a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r a_i \otimes x_i \in M_n(\mathcal{A}).$$

Then for any function  $\varphi$  in  $C_c^\infty(\mathbb{R})$ , we have

$$(i) \quad \mathbb{E}\{(\text{tr}_n \otimes \text{tr}_m)[\varphi(S_n)]\} = \text{tr}_m \otimes \tau[\varphi(s)] + O(n^{-2}).$$

$$(ii) \quad \mathbb{V}\{(\text{tr}_n \otimes \text{tr}_m)[\varphi(S_n)]\} \leq \frac{\pi^2}{8n^2} \|\sum_{i=1}^r a_i\|^2 \mathbb{E}\{(\text{tr}_n \otimes \text{tr}_m)[\varphi'(S_n)^2]\}.$$

**Lemma [Gaussian Poincaré Inequality].** Let  $N$  be a positive integer, let  $\sigma$  be a positive number and equip  $\mathbb{R}^N$  with the probability measure

$$\mu = N(0, \sigma^2) \otimes N(0, \sigma^2) \otimes \cdots \otimes N(0, \sigma^2) \quad (N \text{ terms}).$$

Let, further,  $f: \mathbb{R}^N \rightarrow \mathbb{C}$  be a  $C^1$ -function, such that

$$\mathbb{E}_\mu\{|f^2|\} < \infty.$$

Then

$$\mathbb{V}_\mu\{f\} := \mathbb{E}_\mu\{|f - \mathbb{E}_\mu\{f\}|^2\} \leq \sigma^2 \mathbb{E}_\mu\{\|\text{grad}(f)\|^2\}.$$