Bioinformatics
(Graph Products)

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There are four standard products:

- Cartesian product □
- direct product ×
- strong product ♠
- lexicographic product ○
The vertex set \( V(G_1 \star G_2), \star \in \{\Box, \times, \boxtimes, \circ\} \)

As numbers, one can multiply graphs.

The vertex set \( V(G) \) of the products \( \star \in \{\Box, \times, \boxtimes, \circ\} \) is defined as follows:

\[
V(G_1 \star G_2) = \{ (v_1, v_2) \mid v_1 \in V(G_1), v_2 \in V(G_2) \}
\]
The Cartesian product $G = G_1 \square G_2$

As numbers, one can multiply graphs.

Two vertices $(x_1, x_2), (y_1, y_2)$ in $G$ are linked by an edge if:

1. $[x_1, y_1] \in E(G_1)$ and $x_2 = y_2$ or if
2. $[x_2, y_2] \in E(G_2)$ and $x_1 = y_1$. 

$G_1$

$G_2$
The direct product $G = G_1 \times G_2$

As numbers, one can multiply graphs.

Two vertices $(x_1, x_2), (y_1, y_2)$ in $G$ are linked by an edge if:

1. $[x_1, y_1] \in E(G_1)$ and $[x_2, y_2] \in E(G_2)$.
The strong product $G = G_1 \boxtimes G_2$

As numbers, one can multiply graphs.

Two vertices $(x_1, x_2), (y_1, y_2)$ in $G$ are linked by an edge if:

1. $[x_1, y_1] \in E(G_1)$ and $x_2 = y_2$ or if
2. $[x_2, y_2] \in E(G_2)$ and $x_1 = y_1$ or if
3. $[x_1, y_1] \in E(G_1)$ and $[x_2, y_2] \in E(G_2)$.

$G = G_1 \boxtimes G_2$
The lexicographic product $G = G_1 \circ G_2$

As numbers, one can multiply graphs.

Two vertices $(x_1, x_2), (y_1, y_2)$ in $G$ are linked by an edge if:

1. $[x_1, y_1] \in E(G_1)$ or if
2. $[x_2, y_2] \in E(G_2)$ and $x_1 = y_1$. 

$$G = G_1 \circ G_2$$
Cartesian Product: properties

• commutative
• associative
• distributive w.r.t. disjoint union $+$
• unit element $K_1$, i.e, for all $G$ holds $G \square K_1 \simeq K_1 \square G \simeq G$. 
Cartesian Product: properties

- fiber, layer
- projections (are weak homomorphisms)

**Theorem**

Let \( G = \square_{i=1}^{n} G_i \) and \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in V(G) \). It holds:

\[
d_G(x, y) = \sum_{i=1}^{n} d_{G_i}(x_i, y_i).
\]

**Theorem**

\( G = \square_{i=1}^{n} G_i \) is connected if and only if \( G_i \) is connected for all \( i = 1, \ldots, n \).
Lemma (Square Property)

Let $G = \square_{i=1}^{n} G_i$ be a Cartesian product graph and $e, f \in E(G)$ be two incident edges that are in different fibers. Then there is exactly one diagonal-free square in $G$ containing both $e$ and $f$. 
Prime Factor Decomposition (PFD) w.r.t. $\square$

$G$ is prime, if for all $G_1, G_2$ with

\[ G = G_1 \square G_2 \Rightarrow G_1 \cong K_1 \text{ or } G_2 \cong K_1 \]

**Theorem**

Every connected graph $G = (V, E)$ has a unique representation as a Cartesian product of prime factors (up to isomorphism and the order of the factors).

The number of prime factors is at most $\log_2(|V|)$.

PFD is not unique in the class of disconnected graphs.
Prime Factor Decomposition (PFD) w.r.t. □

**Aim:** Find PFD of given graphs $G$.

**Definition (Product Relation $\sigma$)**

Let $G = □_{i=1}^n G_i$ be a Cartesian product graph. Two edges $e, f$ are in relation $\sigma$, $(e\sigma f)$, if the endpoints of $e$, resp. $f$, differ exactly in the same coordinate $i$.

Thus, for $e$ and $f$ with $e\sigma f$ holds: they are edges of fibers of factor $G_i$. The edges $e$ and $f$ can then be colored with color $i$.

**Aim:** Compute "finest" $\sigma$. 
Djokovic-Winkler-Relation $\Theta$

Two edges $e = (x, y)$, $f = (a, b)$ are in Relation $\Theta$, $(e \Theta f)$, iff

$$d(x, a) + d(y, b) \neq d(x, b) + d(y, a)$$

Lemma

Let $G$ be a graph. It holds:

- For two incident edges $e, f$ holds $e \Theta f$ if and only if $e$ and $f$ belong to a common triangle.
- Let $P$ be a shortest path in $G$ then no two edges of $P$ are in Relation $\Theta$.
- Let $C$ be an isometric cycle of $G$. If $e, f \in E(C)$ are "antipodal" edges then $e \Theta f$.

$\Theta$ is symmetric, reflexive, not transitive.
Djokovic-Winkler-Relation $\Theta$

Two edges $e = (x, y)$, $f = (a, b)$ are in Relation $\Theta$, $(e\Theta f)$, iff

$$d(x, a) + d(y, b) \neq d(x, b) + d(y, a)$$

**Lemma**

*Let e and f be edges of a Cartesian product graph G with $e\Theta f$ then the endvertices of e and f differ in the same coordinate.***

Thus, we can conclude that

$$\Theta \subseteq \sigma.$$ 

**Problem:** even transitive closure $\Theta^*$ is not a product relation. Thus we consider the Relation $\tau$. 
Relation $\tau$

Let $G$ be a graph. Two edges $e = (u, v)$, $f = (u, w)$ are in Relation $\tau$, $(e \tau f)$, iff $e = f$ or $(v, w) \notin E(G)$ and $u$ is the only common neighbor of $v$ and $w$.

**Theorem**

The relation $(\Theta \cup \tau)^*$ is the finest product relation $\sigma$ and thus corresponds to the PFD w.r.t. $\Box$ of a given graph. $(\ldots)^*$ denotes the transitive closure
PFD w.r.t. □

1: **INPUT:** Adjacency-list of a graph $G = (V, E)$
2: Compute equivalences $F_1, \ldots, F_n$ of $(\Theta \cup \tau)^*$
3: **for** $i = 1, \ldots, n$ **do**
4: Compute an arbitrary connected component $G_i$ of $G$ induced by $F_i$
5: Save $G_i$ as prime factor
6: **end for**
7: **OUTPUT:** The prime factors $G_1, \ldots, G_n$ of $G$

**Lemma**

The PFD of $G = (V, E)$ w.r.t. the Cartesian product can be computed in $O(|V||E|)$ time.