A duality approach to Bernoulli convolutions

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December 6th 2013
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1. Bernoulli convolution problem

2. The operators $A^*$ and $A$

3. Spectra and eigenfunctions of $A$

4. Approximation for the density
1. Bernoulli convolution problem
1.1 Bernoulli convolutions

Let \( \frac{1}{2} \leq \lambda < 1 \) and consider the random sum

\[
s_\lambda(\omega) = \sum_{i=0}^{\infty} \omega_i \lambda^i
\]

where \( \omega = \omega_1 \omega_2 \ldots \in \{1, -1\}^\infty \) and

\[
P(\omega_i = 1) = P(\omega_i = -1) = \frac{1}{2}.
\]

What can we say about its distribution?
1.1 Bernoulli convolutions

For $\lambda = \frac{1}{2}$, each random sum

$$\pm 1 \pm \frac{1}{2} \pm \frac{1}{4} \pm \ldots$$

corresponds to a random binary number in $[-2, 2]$. This sum is **uniformly distributed**.
1.1 Bernoulli convolutions

Since $|\lambda| < 1$ we have in the extreme case

$$\sum_{i=0}^{\infty} \lambda^i = \frac{1}{1 - \lambda}$$

so that

$$-\frac{1}{1 - \lambda} \leq s_\lambda(\omega) \leq \frac{1}{1 - \lambda}$$
1.1 Bernoulli convolutions

The **normalized random sum**

\[
S_\lambda(\omega) = \frac{1}{1 - \lambda} \sum_{i=0}^{\infty} \omega_i \lambda^i
\]

takes values in \([-1, 1]\).

We call \(\nu_\lambda\) its probability distribution.
1.2 Some histograms for $\nu_\lambda$

- $\lambda = 0.55$
- $\lambda = 0.618$
- $\lambda = 0.68$
- $\lambda = 0.80$
1.3 Law of pure type

Theorem
(Jessen, Winter, 1935) The measure $\nu_\lambda$ is either singular or has a density with respect to the Lebesgue measure.

For which $\lambda \in [\frac{1}{2}, 1)$ does $\nu_\lambda$ have a density function?
1.4 Pisot numbers

A Pisot number $\beta \in \mathbb{R}$ is a root of a polynomial with leading coefficient 1 and integer coefficients, such that $|\alpha_i| < 1$ for its conjugate roots.

The **golden ratio** $\tau = 1.618...$ is a Pisot number:

$$\tau^2 - \tau - 1 = 0$$
1.5 The singular case

Theorem
(Erdős, 1939) Let $\beta = \lambda^{-1} \in (1, 2]$. If $\beta$ is a Pisot number then $\nu_\lambda$ is singular.

$\nu_\lambda$ for $\beta = \frac{1+\sqrt{5}}{2}$ the golden ratio
1.5 The singular case

The Pisot numbers between 1 and 2 are countable.

\[ p_1 = 1.324..., \] the ‘plastic number’, is the smallest.

Are there more \( \beta \in (1, 2] \) other than Pisot numbers for which \( \nu_\lambda \) is singular?

This question is still open.
1.6 Garsia numbers

A Garsia number $\beta \in \mathbb{R}$ is a root of a polynomial with integer coefficients and constant term $\pm 2$ such that $|\alpha_i| > 1$ for its conjugate roots.

$\sqrt[k]{2}$ are Garsia numbers, as roots of

$$\beta^k - 2 = 0$$
1.7 The absolute continuity case

Theorem
(Garsia, 1962) Let $\beta = \lambda^{-1} \in (1, 2]$. If $\beta$ is a Garsia number, then $\nu_\lambda$ has a bounded density.

\[ \beta = \sqrt{2} \]

\[ \beta = \sqrt[3]{2} \]
1.8 Recent results

Consider the set

$$\{ \lambda \in (\frac{1}{2}, 1) : \nu_{\lambda} \text{ is singular} \}$$

**Theorem**
(Solomyak, 1995) Its Lebesgue measure is 0.

**Theorem**
(Shmerkin, 2013) Its Hausdorff dimension is 0.
2. The operators $A^*$ and $A$
2.1 Self-similarity of $\nu_{\lambda}$

Let $S_{\lambda}$ be the random sum $\sum_{i=0}^{\infty} \omega_i \lambda^i$. Then

$$\mathbb{P}(S_{\lambda} \in E) = \frac{1}{2} \mathbb{P}(S_{\lambda} \in E | \omega_1 = +)$$

$$+ \frac{1}{2} \mathbb{P}(S_{\lambda} \in E | \omega_1 = -)$$

and since $\nu_{\lambda}(E) = \mathbb{P}(S_{\lambda} \in E)$:

$$\nu_{\lambda} = \frac{1}{2} \nu_{\lambda} \circ f_{0}^{-1} + \frac{1}{2} \nu_{\lambda} \circ f_{1}^{-1}$$

with contractions

$$f_{0}(x) = \lambda x + \lambda - 1$$

$$f_{1}(x) = \lambda x - \lambda + 1$$
2.1 Self-similarity of $\nu_\lambda$

\[ \nu_\lambda = \frac{1}{2} \nu_\lambda \circ f_0^{-1} + \frac{1}{2} \nu_\lambda \circ f_1^{-1} \]

with $\beta \in (1, 2]$ and

\[ f_0^{-1}(x) = \beta x + \beta - 1 \]
\[ f_1^{-1}(x) = \beta x - \beta + 1 \]
2.2 The operator $A^*$ associated to $\nu_\lambda$

$A^*$ acts on the space $\mathcal{M}_1$ of measures on $[-1, 1]$

$$A^* \mu = \frac{1}{2} \mu \circ f_0^{-1} + \frac{1}{2} \mu \circ f_1^{-1}$$

with $\beta \in (1, 2]$ and

$$f_0^{-1}(x) = \beta x + \beta - 1$$
$$f_1^{-1}(x) = \beta x - \beta + 1.$$
2.2 The operator $A^*$ associated to $\nu_\lambda$

$\nu_\lambda$ is the **fixed point of** $A^*$, 

$$A^* \nu_\lambda = \nu_\lambda$$

that is,

$\nu_\lambda$ is an **eigenmeasure of** $A^*$ **with eigenvalue 1**
2.3 The operator $\mathbf{A}$

Consider the operator $\mathbf{A}$ acting on the space $\mathcal{C}$ of continuous functions on $[-1, 1]$ 

$$Ah = \frac{1}{2} h \circ f_0 + \frac{1}{2} h \circ f_1$$ 

with $\lambda \in [\frac{1}{2}, 1)$ and 

$$f_0(x) = \lambda x + \lambda - 1$$ 
$$f_1(x) = \lambda x - \lambda + 1.$$
2.3 The operator $A$

It turns out that $A^*$ is the dual operator of $A$ i.e.

\[
\text{Duality } (A^* \mu, h) = (\mu, Ah)
\]

holds for all $\mu \in \mathcal{M}_1$ and $h \in \mathcal{C}$ with the dual pair

\[
(\mu, h) = \int_{-1}^{1} h \, d\mu
\]
3. Spectra and eigenfunctions of $A$
3.1 Spectra and eigenfunctions of $A$

**Theorem**
The operator $A$ has **countably many polynomial eigenfunctions**.
3.2 Polynomial eigenfunctions of $A$

Theorem
For $n \in \mathbb{N}$ the polynomial $p_n : [-1, 1] \to \mathbb{R}$

$$p_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n,k} x^{n-2k}$$

with coefficients given recursively by

$$a_{n,k} = \frac{1}{\lambda^{2k} - 1} \sum_{j=0}^{k-1} \binom{n-2j}{n-2k} (1 - \lambda)^{2k-2j} a_{n,j}$$

for $k \geq 1$ and $a_{n,0} = 1$ else, is an eigenfunction of $A$ with eigenvalue $\lambda^n$. 
3.2 Polynomial eigenfunctions of $A$

Polynomial eigenfunctions $p_1, \ldots, p_6$ for $\lambda = \frac{1}{\sqrt{2}}$
4. Approximation for the density
4.1 Orthogonality

If $\mu$ is an eigenmeasure of $A^*$ and $h$ an eigenfunction of $A$ corresponding to \textbf{different eigenvalues}, then

$$\mu \perp h$$

i.e.

$$(\mu, h) = \int_{-1}^{1} h \, d\mu = 0.$$
4.2 Relation between $\nu_\lambda$ and $p_k$

Since $\nu_\lambda$ has eigenvalue 1, we have

$$\nu_\lambda \perp p_k \quad \text{for} \quad k = 1, 2, \ldots$$

i.e.

$\nu_\lambda$ is orthogonal to all non-constant eigenpolynomials $p_k$
4.3 Duality constraint on the density

Suppose $\nu_\lambda$ has a density $\nu \in \mathcal{L}^2([-1, 1])$. The orthogonality conditions on $\nu_\lambda$ imply

$$\nu \perp p_k \text{ for } k = 1, 2, \ldots$$

with respect to the $\mathcal{L}^2-$ product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx$$
4.4 Construction of the density

We can construct **sequence of polynomial densities** $v_n \in P_n$ by imposing the conditions

$$v_n \perp p_k \quad \text{for} \quad 1 \leq k \leq n$$

or equivalently,

$$\langle v_n, x^k \rangle = m_k \quad \text{for} \quad 1 \leq k \leq n$$

for the $k$—th moments $m_k$ of the density $\nu$. 
4.4 Construction of the density

The coefficients $u$ of the approximation $v_n$ satisfy

$$Hu = (1, 0, \ldots, 0)'$$

where $H \in \mathbb{R}^{(n+1) \times (n+1)}$ is a Hilbert matrix

$$H = \begin{bmatrix}
1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & \ldots \\
0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 & \ldots \\
\frac{1}{3} & 0 & \frac{1}{5} & 0 & \frac{1}{7} & \ldots \\
0 & \frac{1}{5} & 0 & \frac{1}{7} & 0 & \ldots \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}.$$ 

$\Rightarrow$ approximations $v_n$ are unique and symmetric
4.5 Approximating densities for $\lambda = 2^{-\frac{1}{3}}$
4.5 Approximating densities for $\lambda = 0.8$
References

- B. Solomyak. *Notes on Bernoulli convolutions*, Fractal geometry and applications, 2004


- B. Solomyak. *On the random series* \( \sum \pm \lambda^n \), Ann. of Math., 1995

- K. Hare, M. Panju. *Some comments on Garsia numbers*, Math.Comp., 2013

Recursions

Moments of $\nu_\lambda$

$$m_{2k} = - \sum_{i=1}^{k} a_{2k,i} m_{2k-2i}$$

with $m_0 = 1$ and

$$a_{n,i} = \frac{1}{\lambda^{2k-1}} \sum_{j=0}^{i-1} \binom{n-2j}{n-2i} (1 - \lambda)^{2i-2j} a_{n,j}$$

the coefficient of $x^i$ in the eigenpolynomial $p_n$ of $A$ with $a_{n,0} = 1$. 
Recursions

Legendre moments of $\nu_\lambda$

$$(\nu_\lambda, L_{2n}) = \sum_{k=0}^{n} (-1)^{n-k} 4^{-n} \binom{2n + 2k}{2k} \binom{2n}{n + k} m_{2k}$$

where

$$m_{2k} = (\nu_\lambda, x^{2k})$$

are the moments of the convolution measure.
Recursions

Moments of $\nu_\lambda$, another recursion:

$$m_{2k} = \frac{1}{1 - \lambda^{2k}} \sum_{j=0}^{k-1} b_{2k,\lambda}(2j)m_{2j}$$

where $b_{n,\lambda}(\cdot)$ are the weights of the binomial distribution with parameters $n$ and $\lambda$ and $m_0 = 1$