Slices through self-similar sets

Rüdiger Zeller,
joint work with Christoph Bandt,
Greifswald, Germany

March 21, 2013

Bremen Winter School on Multifractals and Number Theory
Finite orbits in branching dynamical systems

Relation to slices through self-similar sets

Slices of finite type

The golden dodecahedron
Finite orbits for multivalued maps

\[ g(x) = 2x \mod 1 \]

All orbits of \( \frac{1}{2} \) are finite, \( g\left(\frac{1}{2}\right) = \{0, 1\} = g^n\left(\frac{1}{2}\right) \).
Finite orbits for multivalued maps

\[ g_1(x) = \beta x, \quad g_2(x) = \beta x + 1 - \beta, \quad \beta = \sqrt{3}. \]
Finite orbits for multivalued maps

\[ g_1(x) = \beta x, \quad g_2(x) = \beta x + 1 - \beta, \quad \beta = \sqrt{3}. \]

The orbit of \( \frac{1}{\beta+1} \) is finite, \( g_1\left(\frac{1}{\beta+1}\right) = \frac{\beta}{\beta+1} \) and \( g_2\left(\frac{\beta}{\beta+1}\right) = \frac{1}{\beta+1}. \)
Definition (BDS)

A branching dynamical system is given by a set of mappings:
\[ B = \{ g_j : l_j \rightarrow l \mid l_j \subset l \subset \mathbb{R}, j = 1, \ldots, k \}. \]
Definition (BDS)

- A branching dynamical system is given by a set of mappings:
  \[ B = \{ g_j : l_j \rightarrow l \mid l_j \subset l \subset \mathbb{R}, j = 1, \ldots, k \} \]

- The set of successors of \( n \)th generation is given by
  \[ B^n(x) = B(B^{n-1}(x)), \text{ where } B(x) = \{ g_j(x) \mid x \in l_j \} \]
Definition (BDS)

- A branching dynamical system is given by a set of mappings: 
  \[ B = \{ g_j : l_j \to I \mid l_j \subset I \subset \mathbb{R}, j = 1, \ldots, k \} \].

- The set of successors of \( n \)th generation is given by 
  \[ B^n(x) = B(B^{n-1}(x)), \text{ where } B(x) = \{ g_j(x) \mid x \in l_j \} \].

- The set of successors produced by \( x \) is given by 
  \[ B^\infty(x) = \bigcup_{n=0}^{\infty} B^n(x) \].
Definition (BDS)

- A branching dynamical system is given by a set of mappings:
  \[ B = \{ g_j : l_j \to l | l_j \subset l \subset \mathbb{R}, j = 1, \ldots, k \} \]

- The set of successors of \( n \)th generation is given by
  \[ B^n(x) = B(B^{n-1}(x)), \quad \text{where} \quad B(x) = \{ g_j(x) | x \in l_j \} \]

- The set of successors produced by \( x \) is given by
  \[ B^\infty(x) = \bigcup_{n=0}^{\infty} B^n(x). \]

When is \( B^\infty(x) \) a finite set?
Finite orbits in branching dynamical systems

\[ g_1(x) = 2x, \quad g_2(x) = 2x - 1, \quad g_3(x) = 2x - \frac{1}{2}. \]
Finite orbits in branching dynamical systems

\[ g_1(x) = 2x, \quad g_2(x) = 2x - 1, \quad g_3(x) = 2x - \frac{1}{2}. \]

\[ B^\infty \left( \frac{1}{3} \right) = \left\{ \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6} \right\} \]
Related work on $\beta$-expansions and Bernoulli convolutions

Study of $\frac{\log |B^n(x)|}{n}$ for $n \to \infty$.

- De-Jun Feng and Nikita Sidorov
  2011 (Monatsh. Math.)
- Simon Baker
  2012 (arXiv: 1208.6195v1)
- Tom Kempton
  2012 (preprint)
Finite orbits in linear branching dynamical systems

Theorem
Let $B = \{g_j : l_j \to l \mid l_j \subset l \subset \mathbb{R}, j = 1, \ldots, k\}$ a BDS with

$$g_j(x) = \beta^{d_j}x + z_j, \quad \beta > 1, \quad d_j \in \mathbb{N}, \quad z_j \in \mathbb{R}.$$
Finite orbits in linear branching dynamical systems

**Theorem**

Let \( B = \{ g_j : l_j \rightarrow I | l_j \subset I \subset \mathbb{R}, j = 1, \ldots, k \} \) a BDS with

\[
g_j(x) = \beta^{d_j}x + z_j, \quad \beta > 1, \ d_j \in \mathbb{N}, \ z_j \in \mathbb{R}.
\]

If

- \( \beta \) is a Pisot number (algebraic integer, whose conjugates are less than 1 in modulus) and
- \( z_j \in \mathbb{Q} (\beta) \),

then \( B^\infty (x) \) is a finite set for all \( x \in \mathbb{Q} (\beta) \).

**Special cases:**

Slices and BDS

A slice is an intersection of a self-similar set and a hyperplane in $\mathbb{R}^n$.

For the hyperplane holds: $H = H(a, \alpha_1, \ldots, \alpha_{n-1})$. 
Slices and BDS
Proposition: Orthogonal slices through Sierpinski gasket.

- Consider the BDS consisting of the mappings
  \[ g_1(x) = 2x, \quad g_2(x) = 2x - 1, \quad g_3(x) = 2x - \frac{1}{2}, \]
  which are surjections on \([0, 1]\),
- and the graph describing the orbit of \(a\).
Proposition: Orthogonal slices through Sierpinski gasket.

- Consider the BDS consisting of the mappings
  \[ g_1(x) = 2x, \quad g_2(x) = 2x - 1, \quad g_3(x) = 2x - \frac{1}{2}, \]
  which are surjections on \([0, 1]\),

- and the graph describing the orbit of \(a\).

If \(B^\infty(a)\) is finite, the graph is the Mauldin-Williams graph of \(h \cap S\).
Slices and branching systems

\[ S_i = f_i(S), \ i = 1, 2, 3. \] The BDS produces intercepts of lines.
Slices and branching systems

\[ S_i = f_i(S), \ i = 1, 2, 3. \] The BDS produces intercepts of lines.
Slices and branching systems

\[ S_i = f_i(S), \ i = 1, 2, 3. \] The BDS produces intercepts of lines.
\[ g(x) = 2x \]
\[ g_1(x) = 2x \]
\[ g_3(x) = 2x - \frac{1}{2} \]
Related work:

- Slicing the Sierpiński Gasket
  Balás Bárány, Andrew Ferguson, Károly Simon
  2011 (preprint)

- Dimension of Slices through the Sierpinski Carpet
  Anthony Manning, Károly Simon
  2010 (appears in TAMS)

- On the Dimensions of Sections through the Graph-directed Sets
  Zhi-Ying Wu, Li-Feng Xi
Slices and BDS

Let the self-similar set $F$ be given by

$$f_j(x) = \frac{1}{\beta_j}(x + v_j), \quad \beta_j > 1, \quad v_j \in \mathbb{R}^n$$

and $H(a, \alpha_1, \ldots, \alpha_{n-1})$ a hyperplane intersecting $F$. 

![Diagram showing a hyperplane intersecting a self-similar set](image)
Slices and BDS

- Let the self-similar set $F$ be given by
  \[ f_j(x) = \frac{1}{\beta_j} (x + v_j), \quad \beta_j > 1, \; v_j \in \mathbb{R}^n \]
- and $H(a, \alpha_1, \ldots, \alpha_{n-1})$ a hyperplane intersecting $F$.

Then the maps of the BDS producing the graph of intersection are given by

\[
g_j(x) = \beta_j x + \langle \begin{pmatrix} -1 \\ \cot \alpha_1 \\ \vdots \\ \cot \alpha_{n-1} \end{pmatrix}, v_j \rangle
\]

and the vertex set is given by $B^\infty(a)$. 
Slices of finite type

Definition
A slice is of finite type if the corresponding graph possesses finite many nodes ($\Leftrightarrow B^\infty(a)$ is a finite set.)
Slices of finite type
Slices of finite type
Proposition (Slices through Sierpinski gasket)

The slice \( g(a, \alpha) \cap S \) is of \textit{finite type}

\[ \iff \]

The numbers \( \frac{\sqrt{3}}{2} \cot \alpha \) and \( a \) are \textit{rational}.
Theorem (Sufficient conditions for Pisot-fractals)

Let $F \subset \mathbb{R}^n$ a self-similar set given by

\[ f_j(x) = \beta^{-d_j}(x + v_j), \quad \text{where } \beta > 1, \ d_j \in \mathbb{N}, \ v_j \in \mathbb{R}^n \]

and let $H(a, \alpha_1, ..., \alpha_{n-1})$ a hyperplane intersecting $F$. 

Assume that the following conditions are fulfilled:

1. $\beta$ is a Pisot number,
2. $\langle \begin{bmatrix} -1 \\ \cot \alpha_1 \\ \vdots \\ \cot \alpha_{n-1} \end{bmatrix}, v_j \rangle \in \mathbb{Q}(\beta)$ for all $j$,
3. $a \in \mathbb{Q}(\beta)$.

Then the slice $H \cap F$ is of finite type.
Theorem (Sufficient conditions for Pisot-fractals)

- Let $F \subset \mathbb{R}^n$ a self-similar set given by
  
  $$f_j(x) = \beta^{-d_j}(x + v_j), \quad \text{where } \beta > 1, \ d_j \in \mathbb{N}, \ v_j \in \mathbb{R}^n$$

- and let $H(a, \alpha_1, \ldots, \alpha_{n-1})$ a hyperplane intersecting $F$.

Assume that the following conditions are fulfilled:

- $\beta$ is a Pisot number,

- $\left\langle \begin{pmatrix} -1 \\ \cot \alpha_1 \\ \vdots \\ \cot \alpha_{n-1} \end{pmatrix}, v_j \right\rangle \in \mathbb{Q}(\beta) \quad \forall j,$

- $a \in \mathbb{Q}(\beta)$.

Then the slice $H \cap F$ is of finite type.
The golden dodecahedron (Mai The Duy, 2011)

50 maps with overlaps,
2 scaling factors

Generated with "IFS Builder 3d v. 1.7.6", A. Kravchenko, D. Mekhontsev, Novosibirsk State University, (C) 1999-2011
The golden dodecahedron
The golden dodecahedron
The golden dodecahedron
The golden dodecahedron
The golden dodecahedron
The golden dodecahedron
The golden dodecahedron
The golden dodecahedron
The golden dodecahedron
Periodic orbits in Bernoulli Convolutions

Bernoulli convolution with slope $\tau = \frac{1+\sqrt{5}}{2}$.

$g_1(x) = \tau x$, $g_2(x) = \tau x + 1 - \tau$
Theorem

If the slope $\beta$ is a Pisot number then all $x \in \mathbb{Q}(\beta)$ possess periodic orbits.
Theorem

If the slope $\beta$ is a Pisot number then all $x \in \mathbb{Q}(\beta)$ possess periodic orbits.

The growth rate of the set $B^n(\frac{1}{2})$ is $3\sqrt{2} \approx 1.26$ (because the successors are doubled every three generations).
Growth rate of $B^n(x)$

We want to approximate the growth of the set $B^n(x)$ so that

$$
\rho^n \approx B^n(x).
$$

for a given constant $\rho$ depending on $x$. 

Growth rate of $B^n(x)$

We want to approximate the growth of the set $B^n(x)$ so that

$$\rho^n \approx B^n(x).$$

for a given constant $\rho$ depending on $x$. This condition is equivalent to

$$\ln \rho \approx \frac{\ln B^n(x)}{n}.$$

We set $\gamma := \ln \rho$. 
Lebesgue typical growth rate in Pisot case

Theorem (Feng, Sidorov (2009))

If the slope $\beta$ is a Pisot number then there exists a constant $\gamma$ so that

$$\lim_{n \to \infty} \frac{\ln B^n(x)}{n} = \gamma$$

for Lebesgue almost every $x \in [0, 1]$, where $\gamma < \ln(2/\beta)$. 

In our case $2/\tau \approx 1.236$. Is $3\sqrt{2} \approx 1.26$ the maximal growth rate?
Lebesgue typical growth rate in Pisot case

Theorem (Feng, Sidorov (2009))

*If the slope $\beta$ is a Pisot number then there exists a constant $\gamma$ so that*

$$
\lim_{n \to \infty} \frac{\ln B^n(x)}{n} = \gamma
$$

*for Lebesgue almost every $x \in [0, 1]$, where $\gamma < \ln(2/\beta)$.*

In our case $2/\tau \approx 1.236$.

Is $3\sqrt{2} \approx 1.26$ the maximal growth rate?
Computersimulation for $B^{16}(x)$

Peak in $x^* \approx 0.4472$. 
Orbit for Peak $x^*$

Some calculations yield:

$$x^* = \frac{2\tau - 1}{5} \approx 0.44721359549996.$$
The growth of $B^n(x^*)$ is dominated by Fibonacci sequence.

$$\Rightarrow \rho_{x^*} = \sqrt{\tau} \approx 1.27$$
Maximal growth rate

Proposition

The maximal growth rate of the Bernoulli convolution with slope $\tau$ is given by $\sqrt{\tau}$. 
Orbit with unusual growth rate

Remark: There exist many orbits with unusual growth rate. One last example:

Every 7 generations $\frac{18-3\tau}{29}$ is produced with multiplicity.
Orbit with unusual growth rate

The multiplicity is $5 \Rightarrow \rho = \sqrt[5]{5} \approx 1.258$