Slices through self-similar sets

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Graph-directed self-similar sets

Cutting plane method

Slices of finite type
Sierpinski-gasket is a graph-directed set.

\[ \Delta = f_1(\Delta) \cup f_2(\Delta) \cup f_3(\Delta) \]

All \( f_j \) are concatenations of scalings and translations in \( \mathbb{R}^2 \).

To construct Sierpinski-gasket we need only a triangular pattern.
Christmas-fractal:
Graph-directed set with several patterns.

The self-similar structure of christmas-fractal consists of several patterns. Can you find them?
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The self-similar structure of christmas-fractal consists of several patterns. Can you find them?
Christmas-fractal

How are the patterns connected in the graph-directed system?
Christmas-fractal

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How are the patterns connected in the graph-directed system?
Equationary system for christmas-fractal

\[ \bigcirc = \bigcup_{i=1}^{6} f_i(\bigcirc) \cup \bigcup_{i=7}^{9} f_i(\Delta) \cup \bigcup_{i=10}^{12} f_i(\nabla) \]

\[ \Delta = \bigcup_{i=13}^{15} f_i(\Delta) \cup f_0(\bigcirc) \]

\[ \nabla = \bigcup_{i=16}^{18} f_i(\nabla) \cup f_0(\bigcirc) \]
Graph-directed sets

A Mauldin Williams-graph is a directed graph \((V,E,i,t,k)\).

\[ k_i(e) \] are indices of mappings contained in an IFS \( \{f_1, \ldots, f_m\} \).
Graph-directed sets

To MW-graph there exists an related equationary system:

\[ A_u = \bigcup_{v \in V} f_{k(e)}(A_v), \quad u \in V, \]

where \( E(u, v) = \{ e \in E \mid i(e) = u, t(e) = v \} \).

\[ \begin{array}{c}
\begin{array}{c}
A_u \\
\end{array}
\end{array} \xrightarrow{k_1(e), \ldots, k_j(e)} \begin{array}{c}
\begin{array}{c}
A_v \\
\end{array}
\end{array} \]
Graph-directed sets

To MW-graph there exists an related equationary system:

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where \( E(u, v) = \{ e \in E \mid i(e) = u, t(e) = v \} \).

Theorem (Existence of graph-directed sets)

If the IFS consists of similtudes, there exist unique, non-empty and compact sets \( A_u \) solving the above equationary system.
Christmas-fractal is a slice through Menger sponge

A slice is an intersection of a self-similar set in $\mathbb{R}^n$ with a hyperplane.

The self-similar structure of a slice might be given by a MW-graph.
How to determine the MW-graph of a slice?
MW-graph of slices

First result: Patterns of graph-directed construction are the four intersections.
MW-graph of slices

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MW-graph of slices

First result: Patterns of graph-directed construction are the four intersections.
\( h \)
\[ g_1(x) = 2x \]
$g_3(x) = 2x - \frac{1}{2}$
Results for orthogonal slices through Sierpinski gasket

- \( g_1(x) = 2x \), \( g_2(x) = 2x - 1 \), \( g_3(x) = 2x - \frac{1}{2} \) must be surjections on \([0, 1]\).
- The graph can be obtained from the underlying dynamical system.
- The method can be applied to every intercept in \([0, 1]\).
Finiteness conditions for BDS

Definition (BDS)
We call a set \( \{g_1, \ldots, g_k\} \) of surjections on an interval \( I \) with real domain a branching dynamical system (BDS).
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When is the graph finite?
Finiteness conditions for BDS

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When is the graph finite?

Theorem (Finiteness conditions for BDS)
Suppose the BDS is given by \(g_j(x) = \beta^{d_j}x + z_j\). Moreover let
- \(\beta\) a Pisot number and
- \(z_j \in \mathbb{Q}(\beta)\).

Then the orbit of \(a\) is finite \(\iff a \in \mathbb{Q}(\beta)\).
Summary on orthogonal slices through Sierpinski gasket

1. From IFS and orthogonal cutting line, the BDS 
   \[ g_1(x) = 2x, \quad g_2(x) = 2x - 1, \quad g_3(x) = 2x - \frac{1}{2} \]
   defined on \([0, 1]\) was derived.

2. The orbit of the intercept produces a graph \(G\).

3. For all rational intercepts, \(G\) is the MW-graph of the slice.

\( h(a, 90^\circ) \)

\( S \)

\( 1/3 \)
Generalisation

- Let the self-similar set $F$ be given by

$$f_j(x) = \frac{1}{\beta_j} (x + v_j), \quad \beta_j > 1, \; v_j \in \mathbb{R}^n$$

- and $H : \langle x, n \rangle = a$ the hyperplane intersecting $F$. 
Generalisation

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1. Then the maps producing the graph of intersection are given by the BDS

$$g_j(x) = \beta_j x - \langle n, v_j \rangle$$

2. and the graph is given by the orbit of $a$. 
Slices of finite type

When is the graph finite?
Slices of finite type

When is the graph finite?

Definition
A slice is of finite type if its graph is finite.
Slices of finite type
Slices of finite type
Theorem (Sufficient conditions for Pisot-fractals)

- Let $F \subset \mathbb{R}^n$ a self-similar set given by
  \[ f_j(x) = \beta^{-d_j}(x + v_j), \quad \text{where } \beta > 1, \ d_j \in \mathbb{N}, \ v_j \in \mathbb{R}^n \]

- and let $H : \langle x, n \rangle = a$ the intersecting hyperplane.
Theorem (Sufficient conditions for Pisot-fractals)

▶ Let $F \subset \mathbb{R}^n$ a self-similar set given by

$$f_j(x) = \beta^{-d_j}(x + v_j), \text{ where } \beta > 1, \ d_j \in \mathbb{N}, \ v_j \in \mathbb{R}^n$$

▶ and let $H : \langle x, n \rangle = a$ the intersecting hyperplane.

Suppose the following conditions to be satisfied:

▶ $\beta$ is a Pisot number,

▶ $\langle n, v_j \rangle \in \mathbb{Q}(\beta) \quad \forall j.$

Then the slice $H \cap F$ is of finite type $\Leftrightarrow a \in \mathbb{Q}(\beta)$. 
Cutting the Menger sponge

Cutting plane method yields BDS with finite orbits:

- IFS given by $f_j(x) = \frac{1}{3}(x + v_j), \ v_j \in \{0, 1, 2\}^3$,
- BDS contains maps $g_j(x) = 3x - \langle v_j, n \rangle$.

Conclusion: All planes $H : \langle x, n \rangle = a$ with $n \in \mathbb{Q}^3$ and $a \in \mathbb{Q}$ produce a finite type slice.
Line segments in Menger sponge
Line segments in Menger sponge

Is it the only line segment?
Line segments in Menger sponge

Projection to coordinate planes are Sierpinski carpets.

All cavaties can be captured by projections to 3 coordinate planes.
Line segments in Menger sponge

What line segments are contained in Sierpinski carpet?
Line segments in Menger sponge

What line segments are contained in Sierpinski carpet?

Proved by Christoph and Mohamed Mubarak
Line segments in Menger sponge

Result:
Beside line segments parallel to edges, there exists only one more contained in Menger sponge.
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Menger sponge – a universal curve

Menger (1926): Menger sponge is a topological one dimensional set. Moreover, of any arbitrarily given curve there exists a "distorted" version contained in Menger sponge.
Golden Dodecahedron

Cutting plane method yields BDS with finite orbits:
- IFS given by $f_j(x) = \tau^{-d_j}(x + v_j)$, $v_j \in \mathbb{Q}(\tau)^3$, $d_j \in \{2, 3\}$,
- BDS contains maps $g_j(x) = \tau^{d_j}x - \langle v_j, n \rangle$.

Conclusion: All planes $H : \langle x, n \rangle = a$ with $n \in \mathbb{Q}(\tau)^3$ and $a \in \mathbb{Q}(\tau)$ produce a finite type slice.
Excavations in Golden Dodecahedron
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Golden Icosahedron

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- BDS contains maps $g_j(x) = \tau^{d_j}x - \langle v_j, n \rangle$.

Conclusion: All planes $H : \langle x, n \rangle = a$ with $n \in \mathbb{Q}(\tau)^3$ and $a \in \mathbb{Q}(\tau)$ produce a finite type slice.