Optimal thickness of a cylindrical shell under dynamical loading

Paul Ziemann

Institute of Mathematics and Computer Science, E.-M.-A. University Greifswald, Germany
1 e-mail paul.ziemann@uni-greifswald.de

Abstract. In this paper an optimization problem for a cylindrical shell under an applied dynamical loading is discussed. The aim is to look for an optimal thickness of a shell to minimize the deformation when the shell is subject to an external time-dependent force together with an initial impact. The volume of the shell has to stay constant during optimization. A model for the deflection is derived and analyzed and a corresponding optimal control problem is formulated. Numerical solutions are presented for a cylindrical panel with a body falling down onto the opposite side of the shell's only clamped edge and then staying there.

Keywords: Optimal control of PDE, Shape optimization, Linear elasticity

1. Introduction

In this paper we discuss an optimization problem in linear elasticity, particularly in shape optimization. The aim is to look for an optimal thickness of a cylindrical shell to minimize the deformation under an initial impact and a (time-dependent) external force, i.e. in a dynamical setup. As an additional restriction, the volume of the shell is prescribed. The deflection is modelled using a generalization of the steady-state “basic shell model” [1] which makes use of the Hypothesis from Mindlin and Reissner but no further simplifications like in Naghdy or Koiter models. Optimization results for steady-state can be found in [2] and for dynamic conditions with rotational symmetric loading in [3].

After stating the deflection model and the optimal control problem we give some analytical results regarding existence and uniqueness of solutions as well as continuity and differentiability of the deflection with respect to the thickness of the shell. We then focus on the numerical solution to an example of a cylindrical panel with one hardclamped edge and a body falling down onto the shell at the opposite side of that edge leading to an initial impact and then acting with its weight force. This problem was formulated by J. Lellep from University of Tartu.
2. Geometrical description of the shell

For the geometrical description, we first need a chart describing the midsurface of the shell. Let \( \omega = [0, l] \times [\varphi_a, \varphi_b] \) be the parameter region and \( z : \omega \rightarrow \mathbb{R}^3, z(x, \varphi) = (x, R \cos \varphi, R \sin \varphi) \) be a mapping to describe the midsurface of a cylindrical panel with length \( l \) and radius \( R \). We call \( S = z(\omega) \) the midsurface of the shell. The vectors \( a^1 = (1, 0, 0) \) and \( a^2 = \frac{1}{R}(0, \sin \varphi, \cos \varphi) \) form a local contravariant basis on \( S \). Additionally, we consider an orthonormal vector \( a^3 = (0, \cos \varphi, \sin \varphi) \). To describe the shell body we introduce \( \tau : S \rightarrow \mathbb{R}^+, \tau \in W^{1,\infty}(S) \) as the thickness of the shell and define the 3D-reference domain

\[
\Omega_{(\tau)} := \left\{ (x, \varphi, h) \in \mathbb{R}^3 \mid (x, \varphi) \in \omega, h \in \left(-\frac{\tau(x, \varphi)}{2}, \frac{\tau(x, \varphi)}{2}\right) \right\}
\]

(1)

together with the mapping \( \Phi_{(\tau)} : \Omega_{(\tau)} \rightarrow \mathbb{R}^3 \)

\[
\Phi_{(\tau)}(x, \varphi, h) = z(x, \varphi) + ha_3 = (0, (R + h) \cos \varphi, (R + h) \sin \varphi).
\]

(2)

We call \( B_{(\tau)} := \Phi_{(\tau)}(\Omega_{(\tau)}) \) the shell body. Let us denote a local covariant basis with \( g_1 = (1, 0, 0), g_2 = (R + h)(0, -\sin \varphi, \cos \varphi) \) and \( g_3 = a_3 \). In the context of shells, we assume \( \tau \) to be small w.r.t. other dimensions, in particular \( \tau(x, \varphi) < R \).

3. Modeling the displacement

We consider a small displacement \( U : B_{(\tau)} \rightarrow \mathbb{R}^3 \) of the shell body. For modeling we use the Reissner-Mindlin kinematical assumptions, which state that normals to the midsurface remain straight and unstretched during deformation. This leads to the displacement ansatz

\[
U(x, \varphi, h) = u(x, \varphi) + h\theta(x, \varphi)
\]

(3)

with \( u = u_1a^1 + u_2a^2 + u_3a_3 \) describing a translation of all points on a line normal to the midsurface in \( z(x, \varphi) \) and \( \theta = \theta_1a^1 + \theta_2a^2 \) representing a rotation vector, all in local coordinates. We introduce the space of admissible displacements

\[
V := \{(u, \theta) \mid u = (u, u_3) \in H^1(S)^2 \times H^1(S), \theta \in H^1(S)^2 \} \cap \mathcal{BC}
\]

(4)

where \( H^1(S) \) and \( H^1(S)^2 \) are Sobolev-spaces for scalar functions and first order tensors on the midsurface, resp. Let us assume for the boundary conditions \( \mathcal{BC} \) that the shell body is hardclamped over the edge \( x = 0 \), i.e. \( (u, \theta)|_{x=0} = 0 \). We next consider the linear 3D-Green-Lagrange-strain tensor which is given by

\[
e_{ij} = \frac{1}{2}(g_i \cdot U,_{j} + g_j \cdot U,_{i}), \quad i, j = 1, 2, 3,
\]

(5)
where $U_{,i}$ means the partial derivative of $U$ w.r.t. to the $i$-th coordinate. Calculation leads to

$$
e := \begin{pmatrix} e_{11} \\ e_{22} \\ \sqrt{2} e_{12} \end{pmatrix} = \begin{pmatrix} u_{1,1} + h\theta_{1,1} \\ u_{2,2} + Ru_3 + h (\theta_{2,2} + \frac{1}{R} u_{2,2} + u_3) + \frac{h^2}{R} \theta_{2,2} \\ \frac{1}{\sqrt{2}} (u_{1,2} + u_{2,1}) + \frac{h}{\sqrt{2}} (\theta_{1,2} + \theta_{2,1} + \frac{1}{R} u_{2,1}) + \frac{h^2}{\sqrt{2R}} \theta_{2,1} \end{pmatrix} \quad (6)$$

and shear strain

$$\zeta = \begin{pmatrix} e_{13} \\ e_{23} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \theta_1 + u_{3,1} \\ \theta_2 + u_{3,2} - \frac{1}{R} u_2 \end{pmatrix} \quad (7)$$

By Hooke’s Law we get a relationship between strains and stresses $\sigma^{ij}$. Adding the assumption that the normal stress $\sigma^{33}$ is zero (plain-stress) and using Voigt-notation we later can represent the equilibrium conditions for isotropic material in a short way with help of the matrices

$$C := \begin{pmatrix} L_1 + 2L_2 & \frac{1}{(R+h)^2} L_1 & 0 \\ \frac{1}{(R+h)^2} L_1 & (L_1 + 2L_2) & 0 \\ 0 & 0 & \frac{2}{(R+h)^2} L_2 \end{pmatrix} \quad \text{and} \quad D := \begin{pmatrix} 4L_2 & 0 \\ 0 & \frac{4}{(R+h)^2} L_2 \end{pmatrix} \quad (8)$$

where $L_1 = \frac{E\nu}{(1+\nu)(1-\nu)}$ and $L_2 = \frac{E}{2(1+\nu)}$ are the 2D-Lamé constants expressed with Young’s modulus $E$ and Poisson’s ratio $\nu$. To account for dynamical loadings and impacts we introduce a time-dependence on $(u(t), \theta(t)) =: y(t), t \in [0,T]$ and assume the deformation to be in the space

$$W(0,T) = \{ y \mid y \in L^2(0,T; V), \partial_t y \in L^2(0,T; \mathcal{H}), \partial_t^2 y(0,T; \mathcal{V}^*) \} \quad (9)$$

where $L^2(0,T; \cdot)$ is a space of abstract functions which are square Bochner-integrable, $\mathcal{H} = L^2(S)^2 \times L^2(S) \times L^2(S)^2$ is a rigged Hilbert space and $\mathcal{V}^*$ is the dual space of $\mathcal{V}$.

By starting from full 3D-equation of momentum $\partial_t^2 U(t) - \text{div}(\sigma(U(t))) = F(t)$ in $\mathcal{B}_{(r)}$ for $t \in [0,T]$ we can first switch to a weak formulation and then simplify using the above mentioned Reissner-Mindlin kinematics and plain-stress assumption, but no further shell-related simplifications, e.g. Naghdy or Koiter models. We can state dynamic deformation equations in variational form:

Given a surface loading $f \in H^1(0,T; L^2(S))$, find $y \in W(0,T)$, so that

$$\rho M_{(r)} \partial_t^2 y(t) + L_{(r)}(y(t)) = F(t) \quad \text{in} \ \mathcal{V}^* \ \text{for all} \ t \in [0,T], \quad (10)$$

together with initial conditions $y(0) = 0, \partial_t y(0) = g/r$ where $g \in \mathcal{V}$ represents an initial impact on the shell body and $\rho$ is the density. In state equation (10)
we have the linear operator $M(\tau) : \mathcal{V}^* \to \mathcal{V}^*$ arising from dynamical part,

$$M(\tau)(u, \theta) = \begin{pmatrix} \tau & 0 & \frac{\tau^3}{12R} \\ 0 & \tau & 0 \\ \frac{\tau^3}{12R} & 0 & \frac{\tau^3}{12} \end{pmatrix} \begin{pmatrix} u \\ u_3 \\ \theta \end{pmatrix}$$  \hspace{1cm} (11)

and the linear operator $L(\tau) : \mathcal{V} \to \mathcal{V}^*$,

$$L(\tau)(u, \theta)(v, \psi) = \int_{\Omega(\tau)} e(u, \theta)^T C e(v, \psi) + \zeta(u, \theta)^T D \zeta(v, \psi) \, dV,$$  \hspace{1cm} (12)

where this part is also known as basic-shell-model, introduced in [1] and analyzed in [2]. The functional $F(t) \in \mathcal{V}^*$ represents a loading applied on the midsurface in orthogonal direction, i.e.

$$F(t)(v, \psi) = \int_{\omega} fv_3 \, dS.$$  \hspace{1cm} (13)

Finally, we can state our optimization problem

$$\min_{\tau \in W^{1,\infty}(S), \; \; y = (u, \theta) \in W(0,T)} J(u, \theta; \tau) := \int_0^T \|u\|^2 \, dt + \frac{\lambda}{2} \|\tau\|_{H^1(S)}^2$$

s.t. $\rho M(\tau) \partial_t^2 y(t) + L(\tau)(y(t)) = F(t)$ in $\mathcal{V}^*$ for $t \in [0,T]$  \hspace{1cm} (14)

$$y(0) = 0, \; \; \partial_t y(0) = \frac{g}{\tau}$$

$$\tau_{\min} \leq \tau(x, \varphi) \leq \tau_{\max} \text{ in } S, \; \; \int_{\omega} \tau \, dS = C$$

4. Analysis of the problem

We want to study state equation (10) according to existence and uniqueness of a solution and Gâteaux-differentiability of the solution w.r.t. thickness. Let us collect all the thickness restrictions in $U_{ad} := \{\tau \in W^{1,\infty}(S) \mid \tau_{\min} \leq \tau(x, \varphi) \leq \tau_{\max} \text{ in } S, \int_{\omega} \tau \, dS = C\}$.

**Theorem 1.** Let $f \in H^1(0,T;L^2(S))$ and $g \in \mathcal{V}$ be given. For $\tau \in U_{ad}$, the variational problem (10) has an unique solution in $W(0,T)$.

**Proof.** Proven by using techniques from [4] together with analysis of $L(\tau)$ from [1] and analysis of $M(\tau)$.
Definition 1. We call the operator $G : U_{ad} \to W(0,T)$, which maps the thickness $\tau$ to the solution $y(\tau) \in W(0,T)$ of (10) control-to-state-operator.

Theorem 2. The operator $G$ is continuous and Gâteaux-differentiable. The directional derivative $G'(\tau)q = \bar{y} = (\bar{u}, \bar{\theta})$ is given as the solution to
\[
\rho M(\tau) \partial_t^2 \bar{y}(t) + L(\tau)(\bar{y}(t)) = -\rho(M'(\tau)q)\partial_t^2 y(\tau)(t) - Z_{q,y(\tau)}(t) \text{ in } V^* \text{ for } t \in [0,T] \\
\bar{y}(0) = 0, \quad \partial_t \bar{y}(0) = -\frac{q}{\tau^2}
\] (15)

where $M'(\tau)q$ means Gâteaux-derivative of $M(\tau)$ w.r.t. $\tau$ in direction $q$ and
\[
Z_{q,u(\tau),\theta(\tau)}(v,\psi) = \\
-\int_\omega \sum_{h_i \in \{-\tau^2,\tau^2\}} \left[ (e(u(\tau),\theta(\tau))^T C e(v,\psi) + \zeta(u(\tau),\theta(\tau))^T D \zeta(v,\psi)) \left( 1 + \frac{h}{R} \right) \right] \frac{q}{2} dS.
\]

Proof. By using results from [2] for $L(\tau)$ and differentiability of $M(\tau)$ w.r.t. $\tau$ together with estimates from theory of hyperbolic PDE regarding dependence of the solution from initial data.

Theorem 3. The reduced objective $J_s : U_{ad} \to \mathbb{R}$, $J_s(\tau) := J(G(\tau),\tau)$ is Gâteaux-differentiable and it holds
\[
J'_s(\tau)q = -\int_0^T \left( \rho(M'(\tau)q)\partial_t^2 y(\tau)(t),p(t) \right) d_H + Z_{q,y(\tau)}(t)p(t) dt + \lambda(\tau,q)_{H^1(S)}
\] (16)

with adjoint $p \in W(0,T)$ as solution to
\[
\rho M(\tau) \partial_t^2 p(t) + L(\tau)(p(t)) = 2(u(\tau)(t),0) \text{ in } V^* \text{ f.a. } t \in [0,T] \\
p(T) = 0, \quad \partial_t p(T) = 0
\] (17)

Proof. By direct calculation.

5. Numerical solution

We solve the reduced problem $\min_{\tau \in U_{ad}} J_s(\tau)$ numerically in Fortran. The spatial discretization is done with 9-node-biquadratic shell elements. For time domain we use a Crank-Nicolson scheme. The resulting system is solved using Pardiso and the optimization is done using IpOpt. Note that the key for a successful optimization process is an expression for the gradient given to IpOpt that is based on (16) and not on finite differences. The optimization starts on a coarse grid consisting of 289 nodes and ends up on a fine grid with about 4000 nodes.
We consider a quarter tube of length 1, i.e. $\omega = [0, 1] \times [\pi/4, 3\pi/4]$ with radius $R = 1$, $E = 210$, $\nu = 0.3$, $\rho = 7.8$ and volume $C = \pi/20$ which corresponds to a constant thickness of 0.1. Minimum and maximum thickness are 0.05 and 0.15, resp. and a small regularization parameter $\lambda \approx 10^{-5}$ is used. We study the time domain $[0, 1]$. A mass “falling” onto the opposite side of the clamped edge $x = 0$ is giving an impact $g(x, \varphi) = -0.1a_3$ for $(x, \varphi) \in [0.9, \pi/2 - 0.05] \times [1, \pi/2 + 0.05]$. The mass stays on the panel and acts with its weight force $f(x, \varphi) = -a_3$ in the same region, see figure 1. The corresponding optimal thickness (solution to (14)) is shown in figure 2. We see a region of maximal thickness surrounding the loaded region as well as kind of trusses emerging to the loose edges of the shell. Like the load, the optimal thickness is symmetric w.r.t. to the $\varphi$-coordinate. Moreover, a small moderate thick region is placed nearer to the clamped edge. This thickness distribution could also be an interesting starting point for topology optimization where regions with minimal thickness are considered as holes.

References