SELF-SIMILAR SETS WITH OPEN SET CONDITION AND GREAT VARIETY OF OVERLAPS

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Abstract. For a very simple family of self-similar sets with two pieces we prove, using a technique of Solomyak, that the intersection of the pieces can be a Cantor set with any dimension in $[0, 0.2]$ as well as a finite set of any cardinality 2^m. The main point is that the open set condition is fulfilled for all these examples.

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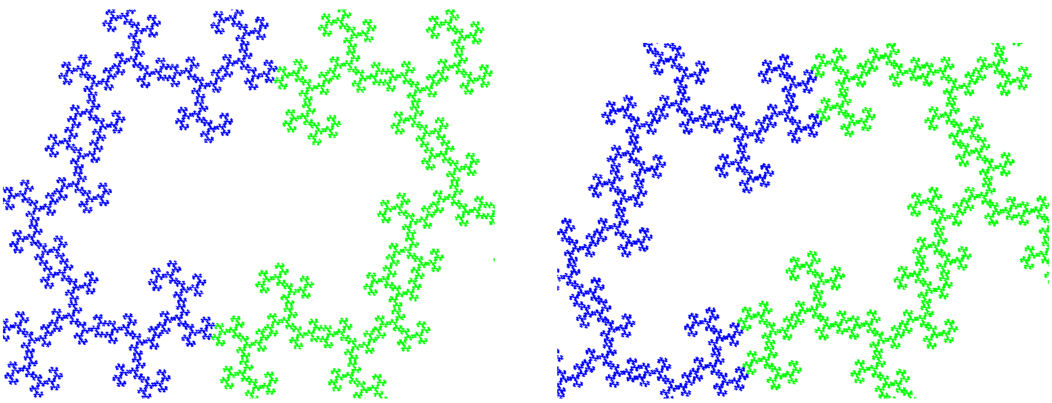

Figure 1. Self-similar sets with Cantor set overlap of dimension around 0.2 on the left, 0.4 on the right.

1. Introduction

Hutchinson [9] gave a precise mathematical framework for self-similarity, the property which was pointed out by Mandelbrot as the essence of fractals. He defined a **self-similar set** A as the unique compact non-empty solution of an equation

$$A = f_1(A) \cup ... \cup f_m(A)$$

where $f_1, ..., f_m$ are given contracting similarity maps on $\mathbb{R}^d$, that is, $|f_i(x) - f_i(y)| = r_i \cdot |x - y|$ for some $r_i < 1$. Since A consists of similar copies $A_i = f_i(A)$ of itself, each A_i consists of smaller copies $A_{ij} = f_i(f_j(A))$, and so on. For any integer n, we can consider the set S^n of words $\mathbf{i} = i_1...i_n$ from the alphabet $S = \{1, ..., m\}$. Writing $f_{\mathbf{i}} = f_{i_1} \cdots f_{i_n}$ and $A_{\mathbf{i}} = f_{\mathbf{i}}(A)$, we have $A = \bigcup\{A_{\mathbf{i}} \mid \mathbf{i} \in S^n\}$. When n tends to infinity, this induces a continuous map $\pi : S^\infty \to A$ from the set S^∞ of sequences $\mathbf{s} = s_1 s_2 s_3...$ onto the self-similar set, the so-called address map. π is defined by

$$\pi(s_1 s_2 s_3...) = \lim_{n\to\infty} f_{s_1} f_{s_2} ... f_{s_n}(x_0) \tag{1}$$

where the choice of $x_0 \in \mathbb{R}^d$ does not matter. See [9, 7, 6] for details.

The geometric structure of A is determined by the overlaps of the A_i. If the contraction factors r_i of the f_i are sufficiently small, the pieces A_i are disjoint, π is a homeomorphism and A a Cantor set. This case is not interesting from a topological viewpoint. For sufficiently large r_i, however, π identifies many addresses, and the overlaps $A_i \cup A_j$ become so big that smaller pieces $A_{\mathbf{i}}$ cannot be recognized. The equation remains formally true, but self-similarity is not realized geometrically.

The proper condition to control overlaps is the **open set condition (OSC)** which was introduced by Moran in 1946 and studied by Hutchinson. Several equivalent formulations of OSC are known [3, 11, 4]. However, it is difficult to verify whether a given set $f_1, ..., f_m$ of similarity maps fulfils OSC or not.

Actually, all examples of self-similar sets with OSC were constructed with rather special methods which either prescribe an open set (e.g. a triangle for the Sierpinski gasket and the von Koch curve) or solve equations for the mappings which in the complex plane results in mappings $f_k(z) = \lambda_k z + v_i$ where the λ_k and v_k are algebraic integers [1, 10]. It seems that it was not known whether there are uncountably many essentially different examples with OSC.

The purpose of the present paper is to show that *there is a great variety of self-similar sets with OSC.* We shall consider a very simple family of fractals $A = A(\lambda)$ in the complex plane with two pieces, given by mappings

$$f_0(z) = \lambda z \qquad , \qquad f_1(z) = \lambda z + 1 \tag{2}$$

where λ is a complex number with $r = |\lambda| < 1$. This family was studied by several authors [6, 12, 13]. We shall only consider λ near to $0.37 + 0.52\,i$. Let the overlap set be $A_0 \cap A_1 = D = D(\lambda)$. Solomyak proved the following theorem.

Theorem 1. *([13], Theorem 2.4) There are uncountably many λ for which the overlap set $D(\lambda)$ is a singleton. The two addresses of this point are different for different λ.*

Together with the statement in [5] that connected self-similar sets in the plane with finite overlap sets fulfil OSC, this provides uncountably many different examples with OSC. In the present note we use Solomyak's technique to give examples with various other overlap sets. Figure 1 shows the type of fractals we shall deal with.

Theorem 2. *For every $m \in \mathbb{N}$ there are uncountably many λ for which OSC holds and the overlap set $D(\lambda)$ consists of 2^m points. The addresses of these points are different for different λ.*

Theorem 3. *For every $\beta \in [0, 0.2]$ there are uncountably many λ for which OSC holds and the overlap set $D(\lambda)$ is a Cantor set of Hausdorff dimension β. The addresses of the points of $D(\lambda)$ are different for different λ.*

2. The structure of overlap

We use the symbol set $S = \{0, 1\}$ for our mappings (2) and we now start addresses $\mathbf{s}$ with s_0 since this simplifies the address map (1) with $x_0 = 0$:

$$\pi(\mathbf{s}) = \pi(s_0 s_1 s_2 ...) = \sum_{k=0}^{\infty} s_k \lambda^k \tag{3}$$

It is easy to check that π is one-to-one if $A_0 \cap A_1 = \emptyset$, and A is connected if $D(\lambda) = A_0 \cap A_1 \neq \emptyset$ [8, 6]. To represent the overlap set, take two sequences $\mathbf{s} = 0s_1s_2...$ and $\mathbf{t} = 1t_1t_2...$ with $\pi(\mathbf{s}) = \pi(\mathbf{t})$. This means

$$\pi(\mathbf{t}) - \pi(\mathbf{s}) = 1 + \sum_{k=1}^{\infty} (t_k - s_k)\lambda^k = 0 \tag{4}$$

Note that $t_k - s_k \in \{-1, 0, 1\}$. Moreover, for any $u_k \in \{-1, 0, 1\}$ there are $t_k, s_k \in \{0, 1\}$ with $u_k = t_k - s_k$. Thus

Remark 1. *$D(\lambda) \neq \emptyset$ if and only if λ is a root of a power series with coefficients -1,0,1.*

When λ is such a root, we can transform the power series into the form

$$f(\lambda) = 1 + \sum_{k=1}^{\infty} u_k \lambda^k \quad , \quad u_k \in \{-1, 0, 1\} \tag{5}$$

and then choose t_k and s_k. If $u_k = 1$ then $t_k = 1, s_k = 0$. If $u_k = -1$ then $t_k = 0, s_k = 1$. However, if $u_k = 0$ then we have two choices: $t_k = s_k = 0$ or $t_k = s_k = 1$.

Notation. We write $\mathbf{u} = 1u_1u_2...$ where $u_{k_j} = 0$ for $j = 1, 2, ...$ In other words, $\mathbf{u} = 1w_10w_20w_30...$ where the $w_j \in \{-1, 1\}^*$ are words made from -1 and +1 (w_j can be empty), and zeros appear at places $k_1, k_2, ...$

If 0 appears at m different places $k_1, ..., k_m$ then we can choose 0 or 1 for each s_{k_j} independently, and we get 2^m possible pairs of addresses $\mathbf{s}, \mathbf{t}$ with $\pi(\mathbf{s}) = \pi(\mathbf{t})$.

Remark 2. *Suppose that λ is the root of exactly one power series of the form (5) with $u_k \in \{-1, 0, 1\}$. Then $D(\lambda)$ is a singleton if no coefficient is 0, and $D(\lambda)$ has 2^m points if just m of the coefficients u_k are zero. If $u_k = 0$ for infinitely many*

$k = k_1, k_2, \ldots$ and $|k_{n+1} - k_n| > 2$ for all sufficiently large n, then $D(\lambda)$ is a Cantor set.

Proof. It remains to verify the last statement. For each w_j in $\mathbf{u}$ there is a word $\tilde{w}_j \in \{0,1\}^*$ which contains the corresponding part of addresses $\mathbf{s}$ associated with $\mathbf{u}$, that is, $\tilde{w}_{ji} = 1$ if $w_{ji} = -1$ and $\tilde{w}_{ji} = 0$ for $w_{ji} = 1$. Thus

$$D = D(\lambda) = \{\pi(\mathbf{s}) \,|\, \mathbf{s} = 0\tilde{w}_1 v_1 \tilde{w}_2 v_2 \tilde{w}_3 v_3 \ldots \text{ with } v_j \in \{0,1\}.\} \tag{6}$$

indicates a binary Cantor structure: $D = D_{0\tilde{w}_1 0} \cup D_{0\tilde{w}_1 1}$, and $D_{0\tilde{w}_1 0} = D_{0\tilde{w}_1 0\tilde{w}_2 0} \cup D_{0\tilde{w}_1 0\tilde{w}_2 1}$ etc., similar as for the whole set A. Since (1) holds for $\pi(\mathbf{s})$, each piece $D_{\tilde{w}}$ is a subset of $A_{\tilde{w}}$.

Is D really a Cantor set? For $\lambda = i/\sqrt{2}$ the set A is a rectangle and D an interval. Thus we have to show that the two subpieces of each piece of D are disjoint. Assume that on the n-th level of D there is some $x \in D_{\tilde{w}0} \cap D_{\tilde{w}1}$. Here $\tilde{w} = 0\tilde{w}_1 v_1 \tilde{w}_2 v_2 \ldots \tilde{w}_n$ has length k_n. Now x has two addresses $\mathbf{s} = \tilde{w}1\tilde{w}_{n+1} v_{n+1} \tilde{w}_{n+2} v_{n+2} \ldots$ and $\mathbf{s}' = \tilde{w}0\tilde{w}_{n+1} v'_{n+1} \tilde{w}_{n+2} v'_{n+2} \ldots$ so that

$$0 = \pi(\mathbf{s}) - \pi(\mathbf{s}') = \lambda^{k_n} + \sum_{m=n+1}^{\infty} (v_m - v'_m)\lambda^{k_m} = \lambda^{k_n}\left(1 + \sum_{m=n+1}^{\infty} (v_m - v'_m)\lambda^{k_m - k_n}\right)$$

Since the coefficients u_k of the power series (5) for λ are unique, this implies that $u_k \neq 0$ is only possible for $k = k_m - k_n, m = n+1, n+2, \ldots$ Thus by our assumption the non-zero elements in $\mathbf{u}$ have distance > 2 for sufficiently large k. Thus there are lots of neighboring zeros which contradicts the same assumption and completes the proof. □

We think the assumption on distance > 2 was not necessary but we wanted to keep the proof short. — In the following we estimate the Hausdorff dimension of the Cantor set D in terms of density of the sequence k_n.

Notation. Let $\mathcal{H}^\alpha$ denote the α-dimensional Hausdorff measure [7], let $a = |A|$ be the diameter of A, and $r = |\lambda|$ the contraction factor of our mappings f_0, f_1.

Remark 3. *If $r^{k_n\gamma} \le c \cdot 2^{-n}$ for a constant c and infinitely many n then $\mathcal{H}^\gamma(D) < \infty$.*

Proof. Take n which fulfils the inequality, and consider the 2^n initial words $\mathbf{s}'$ of length k_n of sequences $\mathbf{s}$ which correspond to $\mathbf{u}$. Then D is covered by the 2^n sets $A_{\mathbf{s}'}$ which all have diameter $r^{k_n}a$. The sum for this covering in the definition of $\mathcal{H}^\gamma(D)$ is

$$2^n \cdot (r^{k_n}a)^\gamma \le 2^n \cdot c \cdot 2^{-n} a^\gamma \le c a^\gamma \ .$$

If this holds for arbitrary large n then $\mathcal{H}^\gamma(D) \le c a^\gamma$. □

Remark 4. *Assume that OSC holds for the set A generated by f_0, f_1, and that the zeros in $\mathbf{u}$ have bounded gaps: $k_{n+1} - k_n < K$ for all n. If $r^{k_n\alpha} \ge c \cdot 2^{-n}$ for a constant c and all $n \ge n^*$ then $\mathcal{H}^\alpha(D) > 0$.*

Proof. Let μ denote the binary construction measure on D, considered as measure on $\mathbb{R}^2$. That is, $\mu(D)=1$ and $\mu(D_w)=2^{-n}$ for each of the 2^n pieces D_w of D which are obtained on the level n of the construction, which corresponds to u_{k_n}.

OSC for A implies that there is a constant L such that each set B of diameter $b=|B|$ hits at most L incomparable pieces A_w of diameter $\geq b\cdot r^K$ [11, 4]. (Instead of r^K we can have any constant, and incomparable means we do not allow words w, w' where w' is an initial subword of w.) For $r^{k_n} < b < r^{k_{n-1}}$ with $n \geq n^*$ this implies

$$\mu(B) \leq 2^{-n}\cdot \mathbf{card}\,\{A_w \mid A_w\cap B\neq\emptyset,\ |w|=k_n\} \leq L2^{-n} \leq \frac{L}{c} r^{k_n\alpha} \leq \frac{L}{c}|B|^\alpha$$

Now take a covering $D\subseteq\bigcup B_i$ where the diameter of each B_i is smaller than $r^{k_{n^*}}$. Then

$$\sum |B_i|^\alpha \geq \frac{c}{L}\sum \mu(B_i) \geq \frac{c}{L}\ .$$

This implies $\mathcal{H}^\alpha(D)\geq \frac{c}{L}$. ☐

3. The main lemma

The parameters λ in our theorems will be defined as roots of power series (5) with special coefficient vectors $\mathbf{u}=u_0u_1...$ The assignment $u\mapsto\lambda$ has a certain continuity property:

Remark 5. *Let f be a power series of the form (5), λ_0 a root of f and G_0 a simply connected neighborhood of λ_0. There is a k such that every power series g of the form (5) with coefficients $v_0v_1v_2...$ such that $v_j=u_j$ for $j\leq k$ has a root in G_0.*

Proof with **Rouché's theorem:** *If f,h are analytic functions on the closure of a simply connected domain G, and for all points z of the boundary ∂G we have $f(z)\neq 0$ and $|h(z)|<|f(z)|$, then f and $f+h$ have the same number of zeros in G.* Since the zeros of f are discrete, we can find a ball $G\subseteq G_0$ around λ_0 such that $f(z)\neq 0$ on $G\setminus\{\lambda_0\}$. Let the minimum of $|f(z)|$ on the boundary ∂G be ε, the radius of the ball δ and $\eta=|\lambda_0|+\delta$. Choose k so that $\eta^k(1-\eta)^{-1}<\varepsilon$, let $h=g-f$ and apply Rouché's theorem. In G every g has exactly one root. ☐

Now we approach the parameters λ which will fulfil our theorems. The set

$$\Lambda=\{\lambda\in\mathbb{C}\mid 0.63<|\lambda|<0.64 \text{ and } 54^\circ<\arg(\lambda)<55^\circ\}$$

contains $\lambda^*\approx 0.3668760+0.5202594i$ which is a zero of the power series $f^*(\lambda)$ with

$$\mathbf{u}^*=u_0^*u_1^*u_2^*...=+-(+++---)^\infty\ .$$

Here we write $'+'$ for 1 and $'-'$ for -1, and $(+++---)^\infty=+++---+++---...$ Direct calculation shows that λ^* is a solution of $1-\lambda+\lambda^2+2\lambda^3=0$, see Example 9.4 in [2]. Solomyak proved Theorem 1 by slightly changing the parameter λ^*.

According to Remark 5, there is a minimal m_0 such that every power series (5) with coefficients starting with $+-(+++---)^{m_0}$ has a zero in Λ. Actually m_0 is not very large: Solomyak notes that he can prove $m_0\leq 14$, and computer experiments indicate that $m_0=2$ [13]. We define

$$\mathcal{B} = \{g(z) = 1 + \sum_{j=1}^{\infty} v_j z^j \mid v_j = u_j \text{ for } j \le 6m_0 + 1, \text{ and } v_2v_3... \text{ is a concatenation of blocks } +++---, ++++---, +++----, ++++----, +++0---\}.$$

For power series f, g of the form (5), the term R_{g-f} will denote a normalized difference: if $g(z) - f(z) = z^l \sum_{j=0}^{\infty} c_j z^j$ with $c_0 \ne 0, l \ge 1$ and $c_j \in \{-2, -1, 0, 1, 2\}$, let

$$R_{g-f}(z) = \frac{g(z) - f(z)}{z^l} = \sum_{j=0}^{\infty} c_j z^j.$$

The following lemma and its proof refine a technique of Solomyak. See Section 4 in [13].

Lemma 1. *For all power series $f \ne g$ of the form (5) with $g \in \mathcal{B}$ and all $\lambda \in \Lambda$ we have*

$$|R_{g-f}(\lambda)| > 10^{-3}.$$

Proof. With l defined as above, we consider an exhaustive list of possibilities for the coefficients $v_l, v_{l+1}, ...$ of $g \in \mathcal{B}$.

(i)	$[v_l, v_{l+5}] = + + + + --$	(i')	$[v_l, v_{l+5}] = - - - - ++$
(ii)	$[v_l, v_{l+4}] = + + + - -$	(ii')	$[v_l, v_{l+4}] = - - - + +$
(iii)	$[v_l, v_{l+3}] = + + --$	(iii')	$[v_l, v_{l+3}] = - - ++$
(iv)	$[v_l, v_{l+3}] = + - --$	(iv')	$[v_l, v_{l+3}] = - + ++$

(v)	$[v_l, v_{l+5}] = + + +0 - -$
(vi)	$[v_l, v_{l+4}] = + + 0 - -$
(vii)	$[v_l, v_{l+4}] = +0 - --$
$(viii)$	$[v_l, v_{l+5}] = 0 - - - ++$

We prove $|R_{g-f}(\lambda)| > 10^{-3}$ by first examining the cases $(i)-(iv)$. Then $(i')-(iv')$ will follow by passing from $R_{g-f}(\lambda)$ to $-R_{g-f}(\lambda)$, and after that we study the remaining cases.

Case (i). From the values of $v_l, ...v_{l+5}$, we derive that $c_0 \in \{1, 2\}, c_1, c_2, c_3 \in \{0, 1, 2\}$, and $c_4, c_5 \in \{0, -1, -2\}$. We can estimate as follows.

$$\begin{cases} Re(c_0\lambda^{-1}) \ge |\lambda|^{-1}\cos(55°) > 0.896, \\ Re(c_1), Re(c_2\lambda), Re(c_4\lambda^3), Re(c_5\lambda^4) \ge 0, \\ Re(c_3\lambda^2) \ge 2|\lambda|^2\cos(110°) > -0.281, \\ Re(\sum_{j=6}^{\infty} c_j\lambda^{j-1}) \ge -\frac{2\cdot|\lambda|^5}{1-|\lambda|} > -0.597. \end{cases}$$

Taking the sum, we get $Re\left(\frac{R_{g-f}(\lambda)}{\lambda}\right) > 0.002$, and

$$|R_{g-f}(f)(\lambda)| = \left|\frac{R_{g-f}(\lambda)}{\lambda}\right| |\lambda| \ge Re\left(\frac{R_{g-f}(\lambda)}{\lambda}\right) |\lambda| > 0.002 \cdot 0.5 = 10^{-3}.$$

Case (ii). We have $c_0 \in \{1, 2\}, c_1, c_2 \in \{0, 1, 2\}$, and $c_3, c_4 \in \{0, -1, -2\}$. Hence,

$$\begin{cases} Re(c_0) \geq 1, \\ Re(c_1\lambda), Re(c_3\lambda^3), Re(c_4\lambda^4) \geq 0, \\ Re(c_2\lambda^2) \geq 2|\lambda|^2\cos(110^\circ) > -0.281, \\ Re(\sum_{j=5}^{\infty} c_j\lambda^j) \geq -\frac{2\cdot|\lambda|^5}{1-|\lambda|} > -0.597. \end{cases}$$

These inequalities lead to $Re(R_{g-f}(\lambda)) > 10^{-3}$. It follows that $|R_{g-f}(\lambda)| > 10^{-3}$.

Case (iii). We have $c_0 \in \{1,2\}, c_1 \in \{0,1,2\}$, $c_2, c_3 \in \{0,-1,-2\}$. Thus,

$$\begin{cases} Re(c_0) \geq 1, \\ Re(c_1\lambda), Re(c_2\lambda^2), Re(c_3\lambda^3) \geq 0, \\ Re(\sum_{j=4}^{\infty} c_j\lambda^j) \geq -\frac{2\cdot|\lambda|^4}{1-|\lambda|} > -0.933. \end{cases}$$

These inequalities also lead to $Re(R_{g-f}(\lambda)) > 10^{-3}$.

Case (iv). We have $c_0 \in \{1,2\}$, $c_1, c_2, c_3 \in \{0,-1,-2\}$. Note that in this case, $v_{l+5} = 1$, and hence $c_5 \in \{0,1,2\}$. So,

$$\begin{cases} Im(-c_0\lambda^{-1}) \geq Im(\lambda)/|\lambda|^2 > 1. \\ Im(-c_1), Im(-c_2\lambda), Im(-c_3\lambda^2), Im(-c_5\lambda^4) \geq 0 \\ Im(-c_4\lambda^3) \geq -2|\lambda|^3\sin(162^\circ) > -0.163. \\ Im(\sum_{j=5}^{\infty} -c_{j+1}\lambda^j) \geq -\frac{2\cdot|\lambda|^5}{1-|\lambda|} > -0.6 \end{cases}$$

It follows that $Im\left(\frac{-R_{g-f}(\lambda)}{\lambda}\right) > 0.002$ and $|R_{g-f}(\lambda)| \geq 0.002 \cdot 0.5$ as in (i).

Case (v). Note that $c_3 \in \{1,0,-1\}$. If $c_3 \in \{0,1\}$, we can use the estimate of (i) while $c_3 = -1$ leads to case (ii).

Case (vi). We have $c_2 \in \{1,0,-1\}$. If $c_2 \in \{0,1\}$, we proceed as in case (ii). Case $c_2 = -1$ leads to case (iii).

Case (vii). $c_1 \in \{0,1\}$ is treated like (iii), and $c_1 = -1$ like (iv).

Case (viii). $c_0 = 1$ leads to (iv) and $c_0 = -1$ to (i'). □

4. Proof of the theorems

Remark 6. *If $g \in \mathcal{B}$ and $\lambda \in \Lambda$ fulfils $g(\lambda) = 0$ then λ is not a root of any other power series f of the form (5).*

Proof. $f(\lambda) = 0$ would impliy $R_{g-f}(\lambda) = 0$ which contradicts our lemma. □

Proof of Theorem 2. We consider all $g \in \mathcal{B}$ with exactly m coefficients 0. By remark 5, to each g we can take a root $\lambda \in \Lambda$. From the definition of $\mathcal{B}$ it is clear that there are uncountably many possibilities, and that the addresses of points in $D(\lambda)$ will be different for different λ. Remark 2 says that $D(\lambda)$ has 2^m points. OSC follows from [5] and also from our next remark. □

Remark 7. *If $g \in \mathcal{B}$ and $\lambda \in \Lambda$ is a root of g then $A(\lambda)$ satisfies the open set condition.*

Proof. According to [3], OSC holds if and only if the identity map is not an accumulation point of the neighbor maps $f_v^{-1} f_w$ where $v = (v_1...v_m), w = (w_1...w_m) \in \{0,1\}^m$ with $v_1 \neq w_1$ and $m \in \mathbb{N}$. In our case, these maps are all translations (cf. [3], Section 2):

$$f_v^{-1} f_w(z) = z + \sum_{j=1}^{m} \frac{1}{\lambda^j} \cdot (w_j - v_j) = z + \frac{p(\lambda)}{\lambda^m}$$

where $p(\lambda) = \sum_{j=0}^{m-1} b_j \lambda^j$ is a polynomial with coefficients $b_j \in \{-1, 0, 1\}$. We need to show that the translation vectors are bounded away from zero, uniformly in p. We can assume that $b_0 = 1$. Since the coefficient of $g(\lambda)$ at either λ^m or λ^{m+1} is nonzero, the lemma gives

$$\left| \frac{p(\lambda)}{\lambda^m} \right| = \left| \frac{g(\lambda) - p(\lambda)}{\lambda^{m+1}} \right| \cdot |\lambda| \geq |R_{g-p}(\lambda)| \cdot |\lambda| > \frac{1}{2} \cdot 10^{-3}. \qquad \square$$

Proof of Theorem 3. We take the $g \in \mathcal{B}$ with infinitely many coefficients 0. For simplicity, we consider only coefficient vectors $+ - w_1 w_2 ...$ where each block w_j is either $+ + + - - -$ or $+ + +0 - - -$. By Remark 6, to each g we can take a root $\lambda \in \Lambda$. Then $D(\lambda)$ is a Cantor set, by Remark 2, and OSC holds by Remark 7. Thus the dimension of $A(\lambda)$ equals $\delta = \frac{\log 2}{-\log r} \geq \frac{\log 2}{-\log 0.63} > 1.5$ where $r = |\lambda|$. In other words, $r = 2^{-1/\delta}$.

Let β be given. Remarks 3 and 4 say that dim $D(\lambda) = \beta$ if $r^{k_n \beta} = 2^{-k_n \beta/\delta}$ is of the same order as 2^{-n}. This means $k_n \approx \frac{\delta}{\beta} \cdot n$ and will be realized by choosing the zero coefficients, i.e. by selecting the w_j. Since $k_{n+1} - k_n \geq 7$, this requires that $\beta < \delta/7$. (With modified blocks including $+ + +0$ and $- - -0$ one can come with β up to $\delta/4$. By the way, periodic patterns of the k_n yield self-similar sets $D(\lambda)$.)

Since r is part of the construction, however, we must determine r and the k_n recursively. We define a decreasing sequence of closed balls $G_l = B_{\eta_l}(\lambda_l)$ where $\eta_l \leq \eta_{l-1}/3$ which will imply the existence of the limit $\lambda = \lim \lambda_l = \bigcap G_l$. We start with $\lambda_0 = \lambda^*$ and $\eta_0 = 0.03$ so that $G_0 \subseteq \Lambda$ since $r_0 \approx 0.6366$. For $\varepsilon = 0.02$ we have $\frac{r_0 + \eta_0}{r_0 - \eta_0} \leq 0.64^{2\varepsilon}$ which implies $(r_0 + \eta_0)^{\beta(1+\varepsilon)} \leq (r_0 - \eta_0)^{\beta(1-\varepsilon)}$ for any β. Moreover, $\left(\frac{r_l + \eta_l}{r_l - \eta_l}\right)^2 < \frac{r_{l-1} + \eta_{l-1}}{r_{l-1} - \eta_{l-1}}$ for $l \geq 1$. (Put $x = \frac{\eta_l}{r_l}, y = \frac{\eta_{l-1}}{r_{l-1}}$, then $x < \frac{1}{3} \frac{0.64}{0.63} y < 0.35y < 0.02$ implies $(1 + x)^2 < 1 + y$ and $(1 - x)^2 > 1 - y$.) Thus

$$(r_l + \eta_l)^{\beta(1+\varepsilon/2^l)} \leq (r_l - \eta_l)^{\beta(1-\varepsilon/2^l)} \qquad \text{for any } \beta \quad \text{and } l = 0, 1, 2, ... \qquad (7)$$

We want to show

$$\beta - \varepsilon/2^{l-1} \leq \dim A(\lambda_l) \leq \beta + \varepsilon/2^{l-1} \quad \text{for } l = 1, 2..., \qquad \text{and } \dim A(\lambda) = \beta. \quad (8)$$

Each λ_l will be a root of a power series $g_l \in \mathcal{B}$ with coefficient vector $\mathbf{v}_l$ of the form $+ - w_1 w_2 ...$ We shall construct an increasing sequence $m_0 < m_1 < m_2 ...$ such that the first m_l blocks w_j are the same for all $\mathbf{v}_k$ with $k \geq l$. The number m_0 corresponding to $\mathbf{v_0} = + - (+ + + - - -)^\infty$ was defined in Section 3, and we have $w_1 = ... = w_{m_0} = + + + - - -$.

Let us define $\mathbf{v}_1$. We take $+++0--- = w_{m_0+1} = w_{m_0+2} = ...$ and note the positions k_1, k_2 of the zeros until for some $n = n_0$, the inequality

$$(r_0 - \eta_0)^{k_n\beta(1-\varepsilon)} \geq 2^{-n} \tag{9}$$

is fulfilled for the first time. Here is the point where we use $\beta \leq 0.2$: For large n we have $k_n < 7.1n$, and $0.63^{(0.2\cdot 7.1)} > \frac{1}{2}$. All k_n for $n > n_0$ are then chosen as large as possible so that (9) remains true. From this maximality and (7) it follows that

$$(r_0 + \eta_0)^{k_n\beta(1+\varepsilon)} \leq c2^{-n} \qquad \text{with } c = 0.64^{6\beta} \geq (r_0 - \eta_0)^{7\beta(1-\varepsilon)}.$$

This way we obtain the sequence $\mathbf{v}_1$, and its power series $g_1 \in \mathcal{B}$ with root $\lambda_1 \in G_0$ such that (8) holds for $l = 1$, by Remarks 3 and 4.

Now we specify our choice of $G_1 = B_{\eta_1}(\lambda_1)$. Besides $\eta_1 \leq \eta_0/3$ and $G_1 \subset G_0$ we require that the power series g_1 associated with $\mathbf{v}_1$ has no other zero in the closure of G_1. Next, m_1 is chosen so that each power series $g \in \mathcal{B}$ which coincides with $\mathbf{v}_1 = +- w_1 w_2 ...$ in the blocks $w_1, ..., w_{m_1}$ must have exactly one zero in G_1. Those blocks are kept for all $\mathbf{v}_k, k \geq 1$.

The k_n which are beyond w_{m_1} are now redefined for $\mathbf{v}_2$. If necessary, we first take blocks $+++0---$ until for some $n = n_1$, the inequality $(r_1 - \eta_1)^{k_n\beta(1-\varepsilon/2)} \geq 2^{-n}$ is fulfilled for the first time. All k_n for $n > n_1$ are then chosen as large as possible so that the inequality remains true. With (7) we have $(r_1 + \eta_1)^{k_n\beta(1+\varepsilon/2)} \leq c2^{-n}$ for $n > n_1$, and we obtain $\mathbf{v}_2, g_2 \in \mathcal{B}$ and $\lambda_2 \in G_1$ such that (8) holds for $l = 2$.

Induction gives $\mathbf{v}_l, \lambda_l, n_l$ for $l \geq 1$. The sequence $\mathbf{v} = \lim \mathbf{v}_l$ corresponds to $\lambda = \lim \lambda_l$. For the k_n in $\mathbf{v}$ holds

$$(r_l - \eta_l)^{k_n\beta(1-\varepsilon/2^l)} \geq 2^{-n} \quad \text{and} \quad (r_l + \eta_l)^{k_n\beta(1+\varepsilon/2^l)} \leq c2^{-n} \quad \text{for } n_l \leq n < n_{l+1}\,.$$

Since $\lambda \in G_l$, the factor $r = |\lambda|$ lies between $r_l - \eta_l$ and $r_l + \eta_l$, the min and max of $|z|$ in G_l, for all l. Thus

$$r^{k_n\beta(1-\varepsilon/2^l)} \geq 2^{-n} \qquad \text{and} \qquad r^{k_n\beta(1+\varepsilon/2^l)} \leq c2^{-n} \qquad \text{for all } n \geq n_l.$$

By Remarks 3 and 4, $\beta(1 - \varepsilon/2^l) \leq \dim A(\lambda) \leq \beta(1 + \varepsilon/2^l)$. This holds for all l, thus $\dim A(\lambda) = \beta$.

In $\mathbf{v}$ there are infinitely many pairs of different blocks w_j, w_{j+1}. Exchanging those two blocks for any set of the pairs gives a new sequence $\mathbf{v}'$ but does not change the dimension since every k_n changes by at most ± 7. So there are uncountably many ways to obtain dimension β. □

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References

[1] C. Bandt, Self-similar measures, *Ergodic Theory, Analysis and Efficient Simulation of Dynamical Systems* (ed. B. Fiedler), Springer 2001, 31–46.

[2] C. Bandt, *On the Mandelbrot set for pairs of linear maps*, Nonlinearity **15** (2002), 1127–1147.

[3] C. Bandt and S. Graf, Self-similar sets VII. A characterization of self-similar fractals with positive Hausdorff measure. *Proc. Amer. Math. Soc.* **114** (1992), 995–1001.

[4] C. Bandt, N.V. Hung and H. Rao, On the open set condition for self-similar fractals, *Proc. Amer. Math. Soc.* **134** (2005), 1369–1374.

[5] C. Bandt and H. Rao, Topology and separation of self-similar fractals in the plane, preprint, Greifswald 2006

[6] M.F. Barnsley, *Fractals Everywhere*, 2nd ed., Academic Press 1993

[7] K.J. Falconer, *Fractal Geometry,* Wiley 1990.

[8] M. Hata, *On the structure of self-similar sets*, Japan J. Appl. Math. **2** (1985), 281–414.

[9] J.E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747.

[10] S.M. Ngai and Y. Wang, Hausdorff dimension of overlapping self-similar sets. *J. London Math. Soc.* **63** (2001), 655–672.

[11] A. Schief, Separation properties for self-similar sets. *Proc. Amer. Math. Soc.* **122** (1994), 111–115.

[12] B. Solomyak, *'Mandelbrot set' for pairs of linear maps: the local geometry*, Analysis in Theory and Applications **20:2** (2004), 149–157.

[13] B. Solomyak, *On the 'Mandelbrot set' for pairs of linear maps: asymptotic self-similarity*, Nonlinearity **18** (2005), 1927–1943.