Global structure from local action: a fractal puzzle and cellular automata

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Abstract

Using the Sierpinski fractal, we show how self-similar hierarchies can develop from local action, either by puzzle pieces or by cellular automata. We do not construct a particular pattern, but uncountably many different patterns, similar as for Penrose tilings. Thus the construction is non-deterministic, and the shape of the final pattern is determined by a sequence of decisions made during construction.

1 Motivation

It is an old question whether harmony of complex structures, which we so frequently observe in nature, comes from a central plan or from self-organization of the parts. When we see a beautiful garden, our first idea is that the chief gardener made an ingenious design and cared for every detail. Political and economical organizations of mankind were always built in a similar hierarchical way, with central forces caring for every detail.

On the other side there is the idea, starting with Adam Smith in economy and Charles Darwin in biology, that a system will develop best when you let all participants act in their own way, according to some general principles. Today, this idea is realized in many fields. Mainframe computers were replaced by networks, and multinational companies organize as networks. Even in the military where efficiency dominates harmony and hierarchy used to be everywhere, networks of individually acting groups have become important.

On an abstract level, however, there is still little understanding on how large and complex structures develop from distributed local action. The main problem seems that the whole system is practically infinite while the locally available information is bounded. How can these tiny pieces of information be passed and composed in order to give a complete description of the system? Here we want to give a small contribution to this problem.

One approach which was very popular among physicists in the early nineties was *self-organized criticality,* including forest-fire and sandpile models based on cellular automata [6]. We shall deal with cellular automata in section 3, but first we follow another line, the *puzzle principle for tilings.* It is quite obvious how the quadratic puzzle piece in figure 1a will generate a tiling of the plane. This tiling is periodic, however. It does not contain infinite information. The famous Penrose puzzle tiles "kite" and "dart" in figure 1b are more exciting. They can be composed to form many different tilings of the plane. None of these tilings is periodic, but all of them show a certain self-similarity. The pieces can be assembled in groups of two or three to form superpieces of similar form, kite and dart, by drawing some auxiliary lines [5, 2]. The superpieces can be assembled again, and so on, and this way one proves that the tilings are non-periodic.

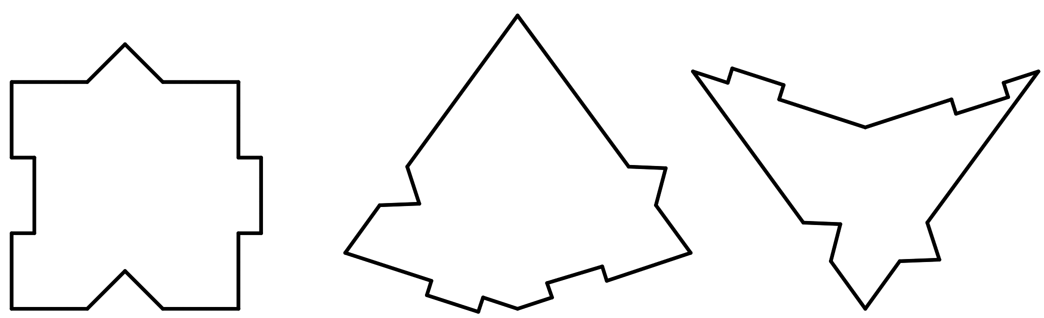

Figure 1. a Periodic tile, b Penrose kite and dart

Interest in aperiodic tiles came from two surprising developments. On one hand, it was proved that the tiling problem is undecidable: There is no algorithm to decide whether a given finite set of tiles will generate a tiling of the whole plane, see [9, 5]. On the other hand, the Penrose puzzle tiles from 1972 were used as models of quasicrystals – non-periodic arrangements of matter with forbidden crystallographic symmetries which since 1984 were found to exist in the form of diverse alloys [10]. From the viewpoint of physics, the puzzle principle seems plausible: atoms assemble due to local forces, and their common action establishes a global structure.

It has turned out that almost all self-similar tilings can be generated by the puzzle principle. Robinson [9], Ammann, Mozes [7], Radin [8] and others gave interesting examples of self-similar tilings generated by local rules, and Goodman-Strauss [4] provided a general and very sophisticated approach for almost arbitrary self-similar tilings. The number of puzzle pieces in the proofs of [8] and [4] can be extremely large, however.

Here we shall consider fractal patterns instead of tilings. This seems to be the first paper where puzzle rules are used to build fractals. From [4] it is not clear whether local rules exist for self-similar patterns with holes.

Actually, we shall restrict our attention to the best-known fractal, the Sierpinski gasket (figure 2) which has been studied in thousands of papers as a tractable prototype of fractal space. Our constructions are easily explained for this example, too. They can be extended to other fractals in a more complicated way.

We consider the Sierpinski gasket as a discrete pattern formed by small atoms. For reasons of symmetry, we now draw the atoms as equilateral triangles. In connection with cellular automata, as in figure 2b below, it is better to take squares. Everywhere, three atoms can be joined to form a larger triangle. On the next level of construction, we proceed with the large triangles as we did before with the small ones. With each level, the radius of action increases. Thus the construction principle is *non-local.*

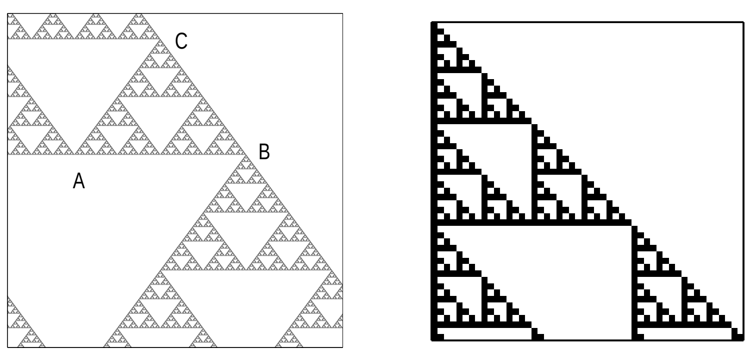

Figure 2. a Some Sierpinski pattern, b generation by cellular automaton

We may consider one particular pattern, where we start with a single atom which is the upper part of a triangle. This triangle is the upper part of a larger triangle, and so on, as indicated in figure 2b. The resulting infinite triangle which extends downwards is well known as the Pascal triangle modulo 2. It is the time development of the linear automaton with rule

$$a'_n = a_{n-1} + a_n \bmod 2$$

(Wolfram's rule 60, cf. [11], p. 25). This would be a local construction of an infinite Sierpinski gasket. We start with a particular point – the center or origin of our pattern – and we obtain one particular infinite pattern which can be described by a finite amount of information. In the following, we shall not consider this particular pattern since it is not complete: the origin atom has only two neighbors, and we can add a third neighbor with another infinite triangle extending upwards to the right (or upwards to the left).

The approach adopted here is more general, however. We study a continuum of different Sierpinski patterns on the plane, in the same way as we have a continuum of Penrose tilings. We start with one atom, which can be considered as either the left, the right or the upper part of a larger triangle. For this larger triangle, as well as for triangles on each successive level, we again have the choice to consider the constructed piece as left, right or upper part of a still larger triangle. Thus an infinite sequence of letters L, R, U will characterize one possible pattern when we refer to the position of our first atom as origin. *This construction is non-deterministic*, resulting in an uncountable number of possible patterns. To construct or identify any of these patterns, we need an *infinite amount of information* – a sequence of letters L, R and U.

Since all patterns contain arbitrary large holes, they can never be periodic. Moreover, this construction is not centralized. From a geometrical viewpoint, there is no property which distinguishes our original atom from any other one. When we start at another position, we get essentially the same patterns but with slightly different sequences. Actually, the mathematical description of the dynamical system of all possible infinite Sierpinski patterns on the plane is a bit intricate (cf. [1]), but we need not go into details here.

2 Puzzle pieces for the Sierpinski gasket

Our first problem is: Can we find puzzle pieces, similar to Penrose's kite and dart, which will exactly produce all Sierpinski patterns? As far as we know, no fractal patterns have been generated so far by the puzzle principle.

Let us investigate the local geometry of the atoms. Each atom has three neighbors. Up to reflection and rotation, all atoms have the neighborhood sketched in figure 3a. They have two neighbors on a basic line, and one neighbor at the top, either right or left. There are three rotated and three reflected versions of this neighborhood of an atom. The cyclic configuration of neighbors shown in figure 3b is not allowed, however. It could be extended, among others, to the periodic pattern of figure 3c.

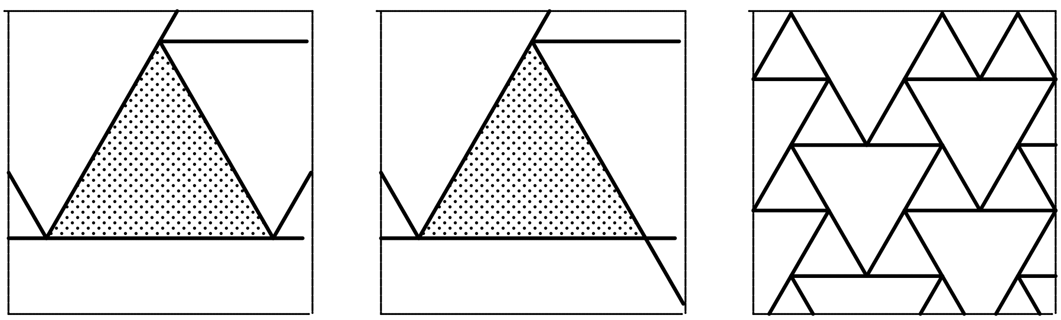

Figure 3. a Sierpinski neighborhood, b cyclic neighborhood, c periodic pattern

We can try to use the six versions of figure 3a as puzzle pieces, with the rule that they join at their corners as prescribed by the neighborhood. However, this cannot prevent a larger triangle consisting of three pieces from being cyclical, as in figure 3b.

Infinite Sierpinski patterns are obtained if and only if triangles with cyclic arrangement of neighbors do not appear on any level of magnification. The construction problem thus consists in breaking the symmetry of triangles of all sizes.

To avoid the cyclic configuration, we need more restrictive rules which imply a symmetry break of the pattern on each possible level of construction. Even for a very large triangle already constructed, when neighbors at vertices A and B are as indicated in figure 2a, this should imply the indicated position of the neighbor at C.

The author admits that for several years, he has doubted that this can be done by local rules. But the solution is rather simple. We store the information concerning the basic line and orientation of any large triangle T_1 along the boundary of its biggest triangular hole. In particular, we can mark that vertex of the hole which is also a vertex of the hole of the triangle T of the next larger generation. And here we can transfer along the lines of this bigger hole the information on the basic line of T_1, as well as of the two other triangles T_2, T_3 surrounding that hole. Our puzzle rules imply that the types of T_1, T_2 and T_3 must fit each other so that the composed triangle is again of admissible type. And the information about the type of T is stored by the puzzle pieces along the boundary of the big hole of T.

Theorem. *There is a set of 10 puzzle pieces, up to rotation and reflection, such that all possible complete patterns composed of these pieces are self-similar Sierpinski patterns, and all Sierpinski patterns can be obtained in this way.*

The puzzle pieces are shown in figure 4, and all of them should be available in rotated and also in reflected form. The **triangle piece** is truncated, and shading indicates its basic line and its orientation at the top vertex. The reflected triangle has other orientation at the top. There are three types of **join pieces** which can connect two neighbor triangles. Moreover, the triangle piece has a slot on that edge which will be inside the next larger triangle, and there are three types of **key pieces** which fit into the slot. Any key will determine a unique ring of three keys and three joins which together with three basic triangles forms a larger triangle. See figure 5. The larger triangle has three ends where joins can be added in the same way as for the basic triangle, and a slot at the midpoint of the appropriate edge, given by the third join piece. The first step of inflation of the pattern is implied by the puzzle pieces.

To apply just the same reasoning on higher levels, indicated in figure 6, we need the three **extension pieces** which provide the connection between keys and joins for the large triangles. We shall see that these pieces cannot be used in the first stage of construction and that they must extend just as far as required.

A pattern will be called complete if no slots, keys or indentations are left open, if each triangle piece is connected with three join pieces, and each join is connected to two triangles. (Physically, the latter conditions could be realized by some other indentations but we did not want to complicate the shapes.) Moreover, we do not allow infinite strips consisting only of extension pieces.

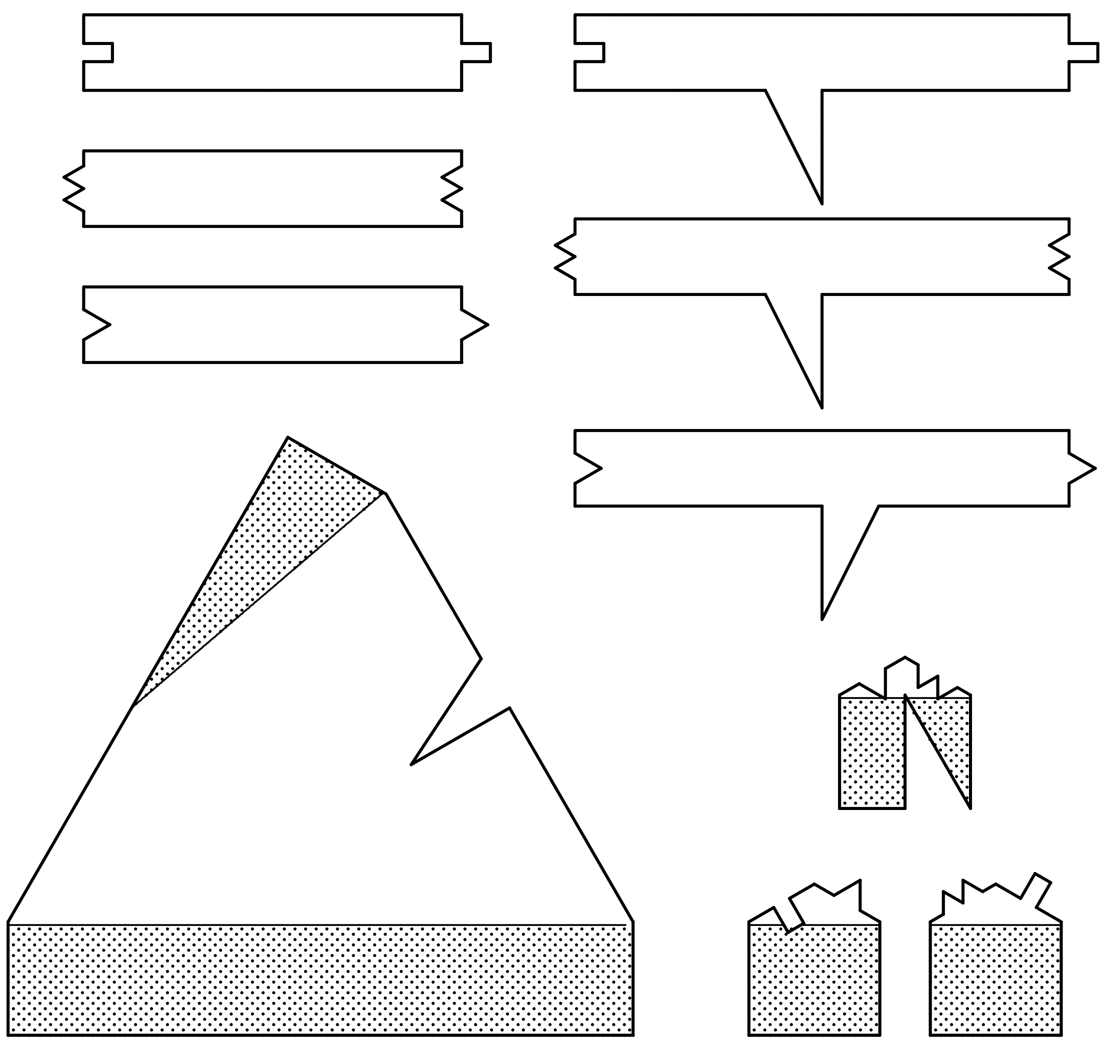

Figure 4. The tiles of our puzzle

The measurements of all pieces are given by two parameters, the side length a of an untruncated triangle, and the length b of the base of a join. The left and right side of a join have length $h = b\sqrt{3}/2$, which is also the height of a slot. The upper sides of the joins, and the width of the keys and extensions is $w = b\sqrt{3}/3$. The length of the extension pieces is $a/2$. The length of the key pieces, and of the slotted side of the triangle piece is $a - 2b$. For the proof we need that a/b is irrational. We have chosen $a = 10$ and $b = \sqrt{3}$ so that $h = 1.5$ and $w = 1$.

Now let us prove the theorem. Suppose we have a complete pattern consisting of copies of our 10 puzzle pieces. All joins and key pieces must be connected to triangles in the obvious way. We show that any given extension piece also has an adjoining triangle. Any chain of extension pieces must lead to a key piece of the same type, which sits in the slot of a triangle. At the end of the triangle there must be a join, which either interrupts the chain, or has its base side in line with the chain. In the second case we trace back the chain by taking the adjoining triangle, the next join piece etc.

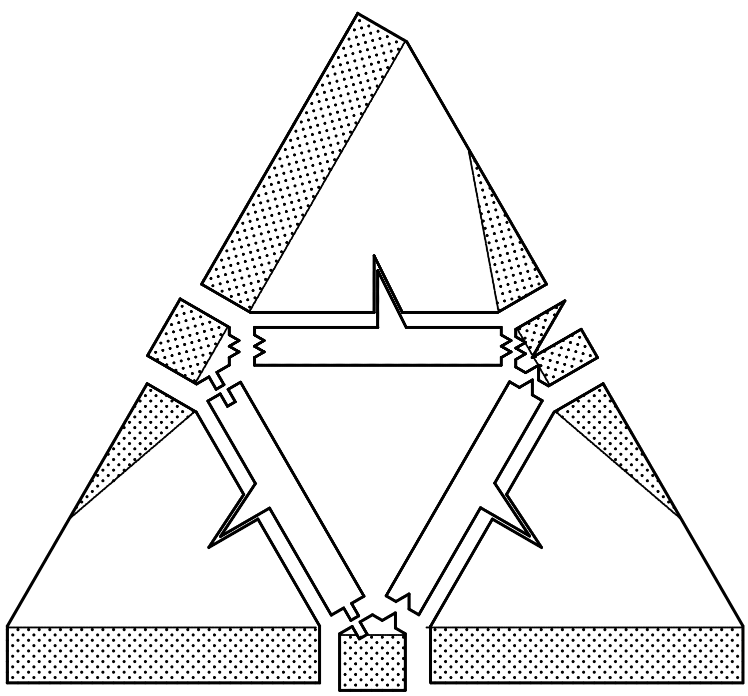

Figure 5. Assembling a larger triangle

Thus in order to see that our complete pattern is of Sierpinski type, we can concentrate on the triangle pieces. It suffices to show that they form a self-similar hierarchy where three triangles assemble to one triangle of the next level. This is done by induction on the level.

For the first level, start with one triangle and add one key piece. All three possible cases (up to reflection and rotation) can be seen in figure 5. It is obvious that the key piece must be closed on both sides by joins, and then it is easy to see that only the configuration of figure 5 can be present in a complete pattern. (If we would continue with extension pieces, the pieces would either cross, or their indentations would remain open.)

So let us take now a triangle of level $k = 2$, and add a key piece. Figure 6 shows that by extension pieces on both sides we reach the endpoints of the slot side of the large triangle, and can add two joins. This also holds for any triangle of higher level k, since the length of the extension pieces is $a/2$. On the other hand, it is not possible to replace a chain of extension pieces by another chain which contains at least one key piece (with the key on the other side) because a/b is irrational.

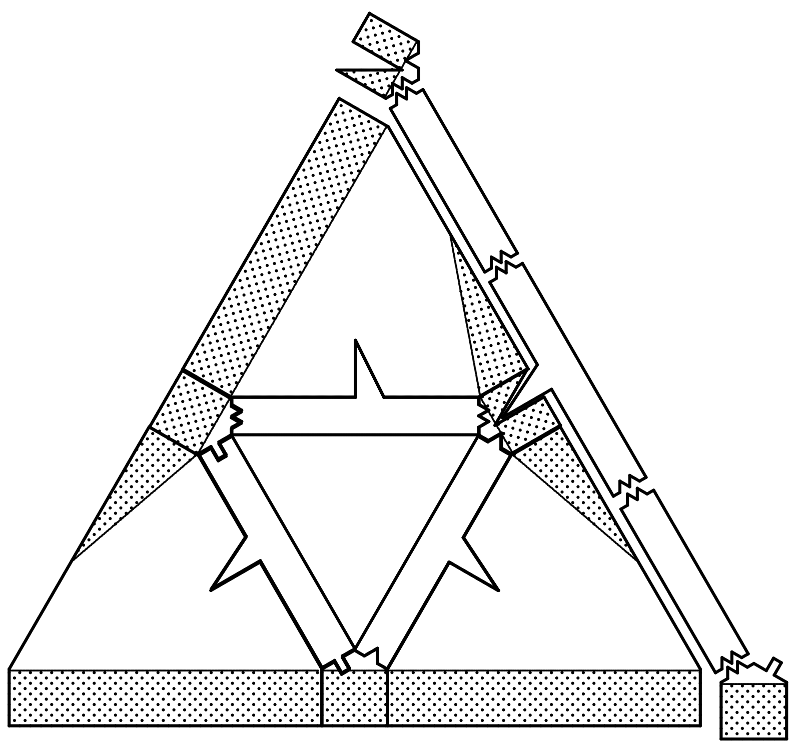

Figure 6. The next steps

Thus we have the arrangement of all pieces of figure 6, or a corresponding arrangement for a triangle of level $k \geq 2$ and its slot side. Let ℓ denote the length of this slot side. Now we continue at the open bindings of the join pieces: we add a triangle and an extension or key piece of the type indicated by the indentation of the join. According to the first argument above, we have a chain of extension pieces and key pieces with adjoining triangles and join pieces, at each of the two open indentations of figure 6. Let ℓ' and ℓ'' denote the lengths of these chains.

If $\ell = \ell' = \ell''$ then we just add one join to complete a "triangle of chains". By the irrational ratio, each of the chains contains exactly one key piece and an adjoining triangle. Now we can use the induction assumption – that the triangle pieces in a complete pattern must assemble with each other and other pieces to form triangles of level k – in order to show that we obtain a triangle of level $k+1$. In this case, the induction step is proved.

The case $\ell' > \ell$ and $\ell'' > \ell$ immediately leads to a contradiction since the two chains added to figure 6 would overlap. The last case to consider is that $\ell' < \ell$. Here we can use the induction assumption to construct a triangle of level $k' < k$ along the whole chain of length ℓ'. Since there is no overlap, the triangle and figure 6 are on different sides of the chain. The chain and the triangle must be closed by a corresponding join piece, and there must be another chain and triangle, with direction turned around 120 degrees. Since there is no overlap, this chain must still be shorter than ℓ' and so the triangle over the chain has level $< k'$. After finitely many steps, such a spiral of smaller and smaller triangles leads to a contradiction, too.

This finishes the proof that complete puzzle patterns must be of Sierpinski type. The reverse statement of the theorem is easily shown.

Remark. We did not attempt to minimize the number of pieces. Coding the key pieces with different lengths rather than different indentations at their ends, and changing the join pieces accordingly, one will need only one type of extension piece, thus reducing the number of pieces to 8. Nevertheless, the author believes that it is impossible to work with less than 5 different pieces.

3 Construction with a cellular automaton

Whoever has assembled puzzle pieces will know that this is made easy by watching the whole picture. Puzzling is probably *not* a local procedure. It is known that some patches of Penrose tiles have a very large "empire" – they determine the arrangement of many further tiles [5]. From the physical viewpoint, this means that the puzzle principle is not as plausible as it seemed to be. Why should some atoms break their bonds because at a distant place no further particles can be attached?

Cellular automata are models of truly local behavior. In this section, we show how all Sierpinski patterns can be constructed by a non-deterministic cellular automaton. Non-determinism will occur only at certain moments and only at one place at a time, and will correspond to the choice of the sequence of letters L, R and U mentioned in section 1. The construction presented below can easily be extended to other self-similar patterns which fit the lattice Z^2, as for instance the chair tiling [10].

We explain the automaton in a similar way as it is done for the synchronization problem in [3], where details on such constructions can be found. We need two algorithms: one which draws larger and larger empty triangles, and one which fills empty triangles of side length 2^n with the Sierpinski pattern. Let us start with the second algorithm which is deterministic and rather simple.

The filling mechanism. We are given a triangle in Z^2 consisting of three finite lines of 2^n cells, one vertical, one horizontal and one diagonal. The signal for filling should arrive at the two middle cells of one of the lines (figure 7a,b). We do not know n, but we can make use of the given pattern.

Because of the recursive structure, it is enough to fill in the three lines of the interior hole, and send a message to their midpoints which indicates that the three smaller outer triangles should be filled by the same algorithm. The algorithm will stop when triangles of side length 2 are reached because then there is no hole.

From the two cells where the algorithm starts (first row in the middle of figure 7), two lines of cells are drawn, as indicated in the upper part of figure 7. When we reach an occupied cell, the lines are reflected from the neighboring cell (second row in the middle) until they meet each other. Now we have found the middle of the side of one of the smaller triangles, and we can call the filling algorithm for this triangle. Beside that, we must send a message to the other two smaller triangles, along the two directions which do not agree with the side which was drawn last. This is demonstrated in the lower part of figure 7.

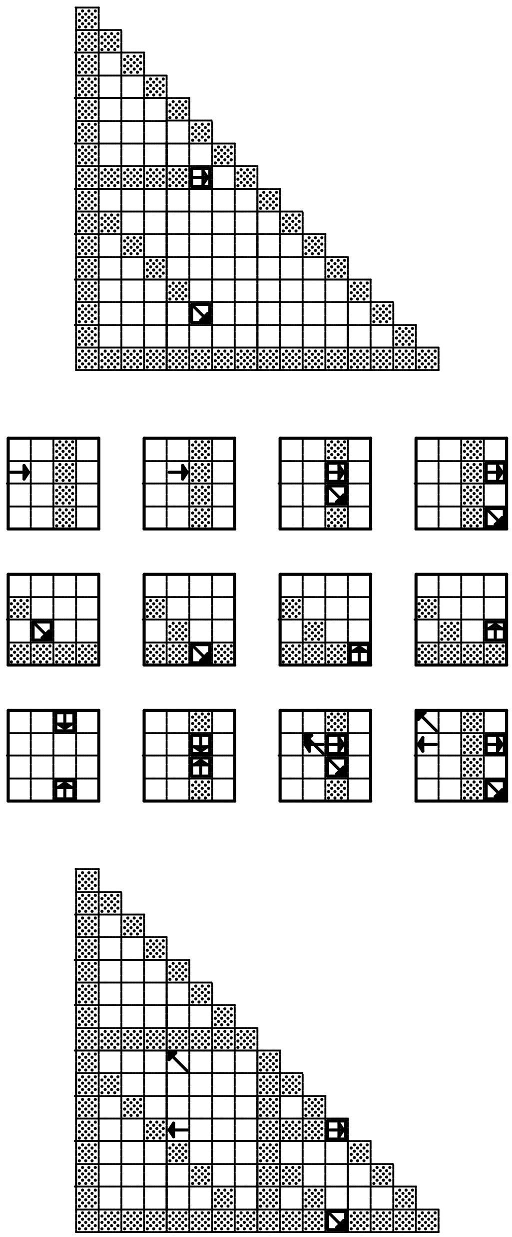

Figure 7. The filling algorithm

The extension mechanism. Now assume we are given a triangle of side length 2^n which is already filled, together with the two middle cells of one side where the extension should start. We shall draw two neighbor triangles of the same size as the given one, such that all three together form a larger triangle with our starting points inside. At the end comes the non-deterministic step: we shall choose one of the sides of the new large triangle and mark the midpoints of this side as starting points for the next step of extension.

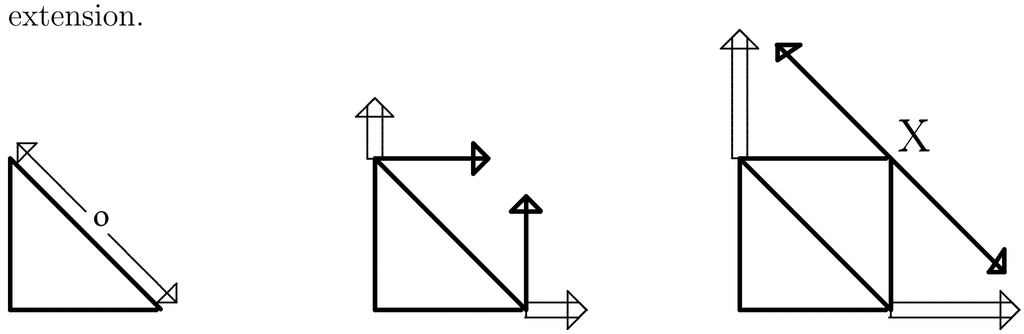

Figure 8. The extension algorithm

The algorithm is illustrated in figure 8. From our starting position o we move into both directions along the side of the triangle until the vertices are reached. At this point we start drawing two lines: one which is reflected as in the filling algorithm, and one which extends that side of the original triangle which we have not come along. The latter line is drawn with half speed, indicated by double arrow in figure 8b.

The two lines drawn with full speed will meet after some steps at X in figure 8c. At this point, they are not extended but reflected so that the last sides of the neighbor triangles are drawn with full speed. They will meet with the other lines drawn with half speed at the vertices of the big triangle, where the drawing just stops. In the meantime, the non-deterministic choice L, R *or* U is done at the point X. If X itself is chosen as the new starting point (in our figure, this is the case L), we can immediately do the next extension step starting from ?. In the other two cases, we have to send a signal from X back along the respective side of the hole.

Finally, we have to care about the filling of the neighbor triangles. One way to do this is to send at the beginning of the extension procedure two signals from our starting point, through the hole as in figure 7d,e, but with half speed. They will meet our full-speed drawn lines at the midpoints of the two other sides of the hole. We can immediately start the filling of the neighbor triangles, but we do it with half speed because the outer sides of the neighbor triangles are not yet complete.

Thus we have shown that all Sierpinski patterns can be constructed by cellular automata as well as by puzzle pieces. For other fractal examples, the midpoints of the sides have to be replaced by appropriate "landmarks" in the basic pattern.

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